Homework 6 solutions.

Problem 14.1. Work out the conjugacy classes of D_5.

Answer. We have that

$$D_5 = \{e, r, r^2, r^4, s, sr, sr^2, sr^3, sr^4\}$$

The conjugacy class of e is just $\{e\}$. To find the conjugacy class of r^n, note that when we conjugate r^n by r^m we get

$$r^m r^n r^{-m} = r^n \quad \forall m$$

and when we conjugate r^n by sr^m we get

$$\begin{aligned} sr^m r^n sr^m &= sr^m sr^{-n} r^m \\ &= ssr^{-m} r^{-n} r^m \\ &= r^{-n} \quad \forall m \end{aligned}$$

(Note that since each reflection sr^m has order 2, $(sr^m)^{-1} = sr^m$.) Thus the conjugacy class of each r^n contains r^n and r^{-n}. In the case of D_5, this gives us the classes

$$\{r, r^4\},\ \{r^2, r^3\}$$

To find the conjugacy class of s, we compute that when we conjugate s by r^m we get

$$r^m sr^{-m} = sr^{-2m} \quad \forall m$$

and when we conjugate s by sr^m we get

$$sr^m ssr^{-m} = s \ \ \forall m$$

Since m can be 1,2,3 or 4, $2m$ is either 2,4,1 or 3 since we have to take $2m$ mod 5. Thus all the reflections form a single conjugacy class

$$\{s, sr, sr^2, sr^3, sr^4\}$$

Since all of the reflections are in this conjugacy class, we don't need to compute the conjugacy class of any of the other reflections. Therefore D_5 has the 4 conjugacy classes listed above. □

Problem 14.4. Calculate the number of different conjugacy classes in S_6 and write down a representative permutation for each class. Find an element $g \in S_6$ such that

$$g(123)(456)g^{-1} = (531)(264)$$

Show that (123)(456) and (531)(264) are conjugate in A_6, but (12345)(678) and (43786)(215) are not conjugate in A_8.

Proof. There is exactly one conjugacy class for each cycle structure of elements in S_6. Thus there are 11 conjugacy types. They can be represented by the following elements

$$e, (12), (123), (1234), (12345), (123456), (12)(34), (123)(45), (1234)(56), (123)(456), (12)(34)(56)$$

We will find a $g \in A_6$ for which $g(123)(456)g^{-1} = (531)(264)$. Note that since (541) and (264) are disjoint, we get that $(531)(264) = (264)(531)$. Since

$$g(123)(456)g^{-1} = (g(1)g(2)g(3))(g(4)g(5)g(6))$$

we will find a g for which $g(1) = 2, g(2) = 6$, $g(3) = 4, g(4) = 5$, $g(5) = 3, g(6) = 1$. Such a g is $(126)(345)$. Since this is a product of two 3-cycles, which are even, g is an even permutation, so $g \in A_6$. Thus $(123)(456)$ and $(531)(264)$ are conjugate in A_6.

Next we will show that $(12345)(678)$ and $(43786)(215)$ are not conjugate in A_8. Note that they are indeed conjugate in S_8. That is, there a g for which

$$\begin{aligned} g(12345)(678)g^{-1} &= \big(g(1)g(2)g(3)g(4)g(5)\big)\big(g(6)g(7)g(8)\big) \\ &= (43786)(215) \end{aligned}$$

For instance, such a g is (14856237). But this g is an 8-cycle, which is not in A_8. Suppose there is some h in A_8 for which they are conjugate. Let $\alpha = (12345)(678)$ and let $\beta = (43786)(215)$. Then $h\alpha h^{-1} = \beta$ and $g\alpha g^{-1} = \beta$. Thus, we get that $h\alpha h^{-1} = g\alpha g^{-1}$. Manipulating this equation, we get that $g^{-1}h\alpha(g^{-1}h)^{-1} = \alpha$. That is, the element $g^{-1}h$ commutes with α. Now if h were in A_8, then h would be an even permutation. Since g is odd, we would get $g^{-1}h$ is an odd permutation. So such an h can only exist if there is an odd permutation that commutes with α.

Suppose there is an odd element f for which

$$\begin{aligned} f(12345)(678)f^{-1} &= \big(f(1)f(2)f(3)f(4)f(5)\big)\big(f(6)f(7)f(8)\big) \\ &= (12345)(678) \end{aligned}$$

Note that $f(12345)(678)f^{-1}$ contains one 3-cycle and one 5-cycle. The 5-cycle must be (12345) and the 3-cycle must then be (678). Thus $f(1)$ must be 1,2,3,4 or 5. If $f(1) = i$ then $f(2)$ must be $i+1$, $f(3)$ must be $i+2$ and so on, where indices are taken mod 5. For instance, if $f(1) = 2$, then $f(2) = 3$ and so on so f contains the 5-cycle (12345). In fact, f must contain a five cycle that is some power of (12345). Likewise, f must contain a 3-cycle that is some power of (678). So f is the product of a power of a three cycle and a power of a five cycle. Since all powers of 3- and 5-cycles are even, f is the product of even permutation. Thus f is even. So the only elements of S_8 that commute with α are even permutation. That is, there are no odd permutations that commute with α. So there can be no odd permutation h for which $g^{-1}h$ commutes with α, since g is even. Therefore α and β are not conjugate in A_8. □

Problem 14.5. Prove that the 3-cycles in A_5 form a single conjugacy class. Find two 5-cycles in A_5 which are not conjugate in A_5.

Proof. To show that the 3-cycles in A_5 form a single conjugacy class, we just need to show that they are all conjugate to (123). That is, for any 3-cycle (abc) we need find a $g \in A_5$ for which $g(abc)g^{-1} = (123)$.

Let $g_0 = (1a)(2b)(3c)$. Then g_0 satisfies the equation $g_0(abc)g_0^{-1} = (123)$. If g_0 is even, we are done. If g_0 is odd, we will find an odd permutation x for which $x(abc)x^{-1} = x$. Then we would get $g_0x(abc)x^{-1}g_0^{-1} = g_0(abc)g_0^{-1} = (123)$. So setting $g = g_0x$ we would note that g is even since g_0, x are both odd, and that $g(abc)g^{-1} = (123)$.

In fact, A_5 permutes 5 numbers, and a, b, c are just three of them. So there are two other numbers d, f in the set $\{1, 2, 3, 4, 5\}$ that are not equal to a, b or c. Thus (abc) and (d, f) are disjoint cycles, so they commute. Then let $x = (df)$. We have that x is an odd permutation, so the above argument holds.

Therefore, for any 3-cycle (abc) we can always find an even permutation g for which $g(abc)g^{-1} = (123)$. So all three cycles are conjugate in A_5.

On the other hand, the 5-cycles (12345) and (12354) are not conjugate in A_5. To see this, let $\alpha = (12345)$ and let $\beta = (12354)$. Then if $g = (45)$ we get that $g\alpha g^{-1} = \beta$. Note that g is an odd permutation, so g is not in A_5. By the same argument we made for Problem 14.4, if h were another permutation for which $h\alpha h^{-1} = \beta$ then the element $g^{-1}h$ would commute with α. Since g is odd, if h were in A_5 then $g^{-1}h$ would be odd.

The only elements that commute with a 5-cycle are powers of that 5-cycle. To see this, suppose we had an f for which

$$\begin{aligned} f(12345)f^{-1} &= \big(f(1)f(2)f(3)f(4)f(5)\big) \\ &= (12345) \end{aligned}$$

Then by the same reasoning as in Problem 14.4, we would get that if f sent 1 to i then f must send 2 to $i+1$ and so on, meaning that f is a power of (12345). In fact, $f = (12345)^{i-1}$. But all powers of (12345) are even permutations. So only even permutations commute with (12345). That means there is no even permutation h for which $h\alpha h^{-1} = \beta$. So α and β are not conjugate in A_5. □

Problem 15.2. Find all normal subgroups of D_4 and D_5.

Proof. Note that if H is a normal subgroup of a group G, and $h \in H$ then the entire conjugacy class of h must be in H. We have already computed the conjugacy classes of D_5 to be $\{e\}, \{r, r^4\}, \{r^2, r^3\}$ and $\{s, sr, sr^2, sr^3, sr^4\}$. To compute the conjugacy classes in D_4, note that the formulae for conjugating rotations and reflections that we computed for D_5 in problem 14.1 hold for D_4 as well. So we get that for rotations r^n,

$$\begin{aligned} r^m r^n r^{-m} &= r^n \quad \forall m \\ sr^m r^n sr^{-m} &= r^{-n} \quad \forall m \end{aligned}$$

and for the reflection s,

$$\begin{aligned} r^m sr^{-m} &= sr^{-2m} \quad \forall m \\ sr^m ssr^{-m} &= s \quad \forall m \end{aligned}$$

So we get that in D_4 the rotations are in a conjugacy class with their inverses giving us $\{r, r^3\}$ and $\{r^2\}$. And we get that s is in a conjugacy class with reflections of the form sr^{2m}. Thus we get the conjugacy class $\{s, sr^2\}$. Doing the above computation for sr we see that in fact sr is conjugate to sr^3. Thus the conjugacy classes of D_4 are

$$\{e\}, \{r, r^3\}, \{r^2\}, \{s, sr^2\}, \{sr, sr^3\}$$

We have already computed all the subgroups of D_4 and D_5 in Homework 2. So the subgroups of D_4 that contain entire conjugacy classes are as follows. Note that these are the normal subgroups of D_4.

$$\{e\},\ <r>=\{e, r, r^2, r^3\},\ <r^2>=\{e, r^2\},\ <r, s>=D_4$$
$$<r^2, s>=\{e, s, r^2, sr^2\},\ <r^2, sr>=\{e, r^2, sr, sr^3\}$$

And the normal subgroups of D_5 are just:

$$\{e\},\ <r>=\{e, r, r^2, r^3, r^4\},\ <r, s>=D_5$$

□

Problem 15.4. Is O_n a normal subgroup of $GL_n(\mathbb{R})$?

Answer. No. For example, when $n = 2$, let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

Then A is in O_2 while B is just in $GL_2(\mathbb{R})$. We will compute BAB^{-1} and show it is not in O_2. So,

$$B^{-1} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}$$

giving us

$$BAB^{-1} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}$$

which simplifies to the matrix

$$C = BAB^{-1} = \begin{pmatrix} 1 & 2 \\ 1/2 & 1 \end{pmatrix}$$

Note that the top left entry of $C^T C$ is $1\frac{1}{4}$, so $C^T C$ is not the identity.

For an arbitrary n, we can modify the matrices A and B slightly to get a counterexample. In fact, all we need to do is append the $(n-2) \times (n-2)$ identity matrix to A and B to get matrices

$$A_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \quad B_n = \begin{pmatrix} 2 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

where the top left corners are the matrices A and B and the bottom right corners contain the $(n-2) \times (n-2)$ identity matrix. Note that A_n is still in O_n. The inverse of B_n is just the matrix formed in the same way with B^{-1} in the top left corner and the $(n-2) \times (n-2)$ identity matrix in the bottom right. And taking $B_n A_n B_n^{-1}$ just gives us, again, the $n \times n$ matrix with BAB^{-1} in the top left corner and the $(n-2) \times (n-2)$ identity matrix in the bottom right. Since BAB^{-1} is not in O_2, the matrix $B_n A_n B_n^{-1}$ is not in O_n. Thus O_n is not a normal subgroup of $GL_n(\mathbb{R})$. □

Problem 15.6. If H, J are normal subgroups of a group, and if they have only the identity element in common, show that $xy = yx$ for all $x \in H, y \in J$.

Proof. Suppose H, J are normal subgroups of a group G and that $H \cap J = \{e\}$. Suppose $x \in H$ and $y \in J$. Consider the element $xyx^{-1}y^{-1}$ in G. Note that xyx^{-1} is an element of xJx^{-1}. Since J is normal, we know that $xJx^{-1} = J$, so in fact xyx^{-1} is an element of J. Since y^{-1} is also an element of J (since J is a subgroup of G), we have that actually $xyx^{-1}y^{-1}$ is an element of J.

By the same reasoning, we see that $yx^{-1}y^{-1}$ is an element of H. Thus $xyx^{-1}y^{-1}$ is also an element of H. Since it is an element of both groups, and H and J only have the identity in common, we must have that $xyx^{-1}y^{-1} = e$. But this is the same as saying that $xy = yx$ for every x in H and y in J. □