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On the topology of orbits for actions of algebraic groups and related results

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Abstract

We investigate the problem of closedness of (relative) orbits for the action of algebraic groups on affine varieties defined over complete fields, its relation with the problem of equipping a topology on cohomology groups (sets), and some estimates for relative orbits of subsets. This is joint work with Nguyen Quoc Thang.

Set up

(1) Let k be a local field. For any affine k-variety X, we can endow $X(k)$ with the (Hausdorff) v-adic topology induced from that of k.

(2) $x \in X(k)$, we are interested in a connection between the Zariski-closedness of $G.x$ and Hausdorff closedness of $G(k).x$.

(3) $k = \mathbb{R}, \mathbb{Q}_p$: considered by Borel/Harish-Chandra (1962), D. Birkes (1971), P. Slodowy (1989). Namely,
If G is a reductive group over $k = \mathbb{R}$ or $\mathfrak{p}$-adic field, $G.x$ is Zariski closed if and only if $G(k).x$ is closed.

Set up (continued)

(4) *Aim:* What extent the above result still hold for more general class of algebraic groups and complete fields?

(5) The above problem relates to the problem of equipping a topology on cohomology groups (or sets), which has important aspects, say in duality theory for Galois or flat cohomology in general (J. Milne (1986), S. Shatz (1964, 1972)).

(6) Note that in the case char. $k = p > 0$, the stabilizer G_v is not smooth in general. Therefore, it needs some careful considerations.

Theorem (B.-N.Q.Thang 2013)

Assume that k is local field, G a smooth affine k-group, acting k-regularly on an affine k-variety V, $v \in V(k)$.

(1) **(a)** $G(k).v$ is closed in $(G.v)(k)$. Thus if $G.v$ is closed in V, then $G(k).v$ is Hausdorff closed in $V(k)$.

(b) If moreover G_v is smooth over k, then for any $w \in (G.v)(k)$ the relative orbit $G(k).w$ is open and closed in Hausdorff topology of $(G.v)(k)$.

(2) Assume that $G(k).v$ is Hausdorff closed in $V(k)$. Then if either

(a) G is nilpotent, or

(b) G is reductive and the action of G is strongly separable,

then $G.v$ is Zariski closed in V.

(3) Assume further that k is perfect, $G = L \times_k U$, where L is reductive, U is unipotent defined over k, then $G(k).v$ is closed if and only if $G.v$ is Zariski closed.

Notion.

The action of G is said to be strongly separable (after Ramanan-Ramanathan) at v if for all $x \in cl(G.v)$, the stablizer G_x is smooth, or equivalently, $G \to G/G_x$ is separable.

Proposition 1. (compare with Borel-Tits, Bremigan,Gille/Moret-Bailly)
If G_v is a smooth k-subgroup of G, then for any $w \in (G.v)(k)$, the relative orbit $G(k).w$ is open and closed in $(G.v)(k)$.

First proof. (Basically due to Borel-Tits (1965)) G_v is smooth $\Rightarrow$ for all $w \in (G.v)(k)$, the projection

$$\pi' : G \to G.w = G.v,$$
$$g \mapsto g.w,$$

is also separable and defined over k. Therefore, $\pi'_k : G(k) \to (G.w)(k)$ also has surjective differential, so it is open by Implicit Function Theorem. Thus all $G(k)$-orbits $G(k).w$ are open and then they are also closed in $(G.v)(k)$.

Remark

In the case that G_v need not be smooth, the closedness of relative orbits still holds, while the openess may fail. Namely, let $G := \mathbb{G}_a \curvearrowright \mathbb{A}^1, g.x = g^p x$. Then for $x = 1$, $G(k).x = k^p$ is closed but not open in k.

Indeed, assume that $k = k_0((T))$, $k_0 = \mathbb{F}_q$. Consider $u_n = 1 + t^p + t^{p^2} + \cdots + t^{p^n} + t^{p^n+1} \notin k^p$.

Hence, $u_n \to u = 1 + t^p + t^{p^2} + t^{p^3} + \cdots \in k^p$. So $k \setminus k^p$ is not closed. Therefore, k^p is not open in k.

Theorem

$G(k).v$ is closed in $(G.v)(k)$.

Remark

Bate-Martin-Roerhle-Tange (2013) consider the topology on $V(k)$ induced from Zariski topology and show the codition that $G.v$ is closed does not imply $G(k).v$ is closed. Namely, let $k = \mathbb{R}$, $G = \mathrm{SL}_2$ acting on $V = G$ by conjugation, and let

$$v = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, g = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Then $v, w \in V(\mathbb{R})$, and $w = g.v$, so v, w are $G(\mathbb{C})$-conjugate but not $G(\mathbb{R})$-conjugate. Hence, w lies in the closure in $V(\mathbb{R})$ of $G(\mathbb{R}).v$, which implies that $G(\mathbb{R}).v$ is not closed in $V(\mathbb{R})$ (in the above topology induced from Zariski topology). But $G.v$ is closed since v is semisimple.

Idea of the proof

(1) We can claim that if $\gamma : G \to H$ is a k-morphism of algebraic groups, then the image $\gamma_k(G(k))$ is closed in $H(k)$.

(2) For each scheme of finite type X over k, we let F be the Frobenius map and $X^{(p^n)} = X \times_k k^{(p^n)}$, $F^n : X \to X^{(p^n)}$, ${}_F X = \mathrm{Ker} F^n$. If $X = G$, then $H = G/{}_F G$ is smooth affine k-group for n sufficiently large.

(3) From the equality $F^n(X(k)) = X^{(p^n)}(k^{p^n})$, and by using the Frobenius power, we can reduce to the case that $G \xrightarrow{\gamma} H$ has smooth kernel. By Implicit Function Theorem, we get the conclusion.

Zariski closed orbits for actions of algebraic groups over arbitrary complete fields

Proposition

Let k be a local field, G a smooth affine group scheme of finite type acting k-regularly on an affine k-variety V. Let $v \in V(k)$ be a k-point. Assume that $G(k).v$ is closed in Hausdorff topology induced from $V(k)$. Then $G.v$ is closed (in Zariski topology) in V in either of the following cases:

(1) G is nilpotent.

(2) G is reductive and the action of G is strongly separable at v in the sense of Ramanan-Ramanathan, i.e. for all $x \in cl(G.v)$, the stabilizer G_x is smooth, or equivalently, the induced morphism $G \to G/G_x$ is separable.

Notion

Let $f : \mathbb{G}_m \to V$ be a morphism of algebraic varieties. If f can be extended to a morphism $\tilde{f} : \mathbb{G}_a \to V$, with $\tilde{f}(0) = v$, then we write $\lim_{t\to 0} f(t) = v$.

Idea of the proof

Assume that $G.v$ is not closed, i.e. $cl(G.v) \setminus G.v \neq \emptyset$. If G is nilpotent (resp., G is reductive with the action is strongly separable), by using Birkes (resp., Ramanan-Ramanathan), there exists $\lambda : \mathbb{G}_m \to G$ defined over k such that $\lambda(t).v \to y \in Y$ while $t \to 0$. Thus, $y \in \overline{G(k).v} \setminus G.v$. Hence, $G(k).v$ is not closed, and we get a contradiction.

Remarks

Generally, G_v needs not be smooth. There are some counter-examples to the effect that in any characteristic, if one of the conditions on G in the above Proposition (i.e., nilpotency, or the strong separability of the action), is removed, then the assertion does not hold. It does not hold already for $G = \mathrm{SL}_2$, if one removes some hypotheses on strong separability.

Example

Let p be a prime, $k = \mathbb{F}_q((T))$, $q = p^r$, $G = \mathrm{SL}_2$, B the Borel subgroup of G consisting of upper triangle matrices, and let ρ the representation of G into 2-dimensional k-vector space V given by

$$\rho : G = \mathrm{SL}_2 \to \mathrm{GL}_2, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}, v = (1, T) \in V(k)$$

Then

(1) $G.v = V \setminus \{(0,0)\}$ is open (and not closed) in the Zariski topology in V and $G(k).v$ is closed in the Hausdorff topology in $V(k)$.

(2) $B.v = \{(x,y) \in V \mid y \neq 0\}$ is open (and not closed) in the Zariski topology in V and $B(k).v$ is closed in the Hausdorff topology in $V(k)$.

Example

Assume further that k is a local field of characteristic 0, $G = \mathrm{SL}_2$, B the Borel subgroup of G, consisting of upper triangular matrices. Consider the standard representation of G by letting G act on the space $V_2 = \bar{k}[X, Y]_2$, considered as 3-dimensional $\bar{k}$-vector space with the canonical basis $\{X^2, XY, Y^2\}$. Then for $v = (1, 0, 1) \in V_2$, we have

(1) $B.v = \{(x, y, z) \mid 4xz = y^2 + 4\} \setminus \{z = 0\}$ is not Zariski closed;

(2) $B(k).v = \{(a^2 + b^2, 2bd, d^2) \mid ad = 1, a, b, c, d \in k\}$ is closed in the Hausdorff topology, where k is either $\mathbb{R}$ or a $\mathfrak{p}$-adic field, with $p = 2$ or $p \equiv 3 \pmod 4$. Moreover, if we set $n := [k^\times : k^{\times 2}]$, then we have the following decomposition $(B.v)(k) = \cup_{1 \le i \le n} e_i(B(k).v))$, where e_i are different representatives of cosets $k^\times$ modulo $k^{\times 2}$, thus $(B.v)(k)$ is also closed in the Hausdorff topology.

Zariski closed orbits for actions of algebraic groups over perfect complete fields

Theorem (An extension of a theorem of Kempf)

Let k be a perfect field, $G = L \times U$, where L is reductive and U is a smooth unipotent k-group. Let G act k-regularly on an affine k-variety V, and let v be a point of instability of $V(k)$, i.e. the orbit $G.v$ is not closed. Let Y be any closed G-invariant subset of $cl(G.v) \setminus G.v$. Then there exist a one-parameter subgroup $\lambda : \mathbb{G}_m \to G$, defined over k and a point $y \in Y \cap V(k)$, such that $\lim_{t \to 0} \lambda(t).v = y$.

Remark

In fact, in the reductive case, original theorem of G. Kempf gives more information about the nature of instable orbits. A version of Kempf's Theorem is originated from the works of Hilbert-Mumford-Birkes-Raghunathan.
Namely, in the case that the field k is an algebraically closed field **(Hilbert-Mumford)**, $\mathbb{R}$ **(Birkes)**, number field **(Raghunathan)**, perfect field **(Kempf)**.

Theorem (Hilbert-Mumford-Birkes-Raghunathan-Kempf)

Let k be a perfect field, G a reductive group defined over k. Let G act k-regularly on an affine k-variety V, and let v be an unstable point $V(k)$, i.e. $0 \in \overline{G.v}$. Then there exist a one-parameter subgroup $\lambda : \mathbb{G}_m \to G$, defined over k such that $\lim_{t \to 0} \lambda(t).v = 0$.

Corollary

Let k be a perfect, local field.

(1) If $G = L \times U$, then $G.v$ is closed if and only if $G(k).v$ is Hausdorff closed.

(2) Let G be a smooth nilpotent k-group, T the unique maximal k-torus of G. Then TFAE:

- **(a)** $G.v$ is closed.
- **(b)** $T.v$ is closed.
- **(c)** $G(k).v$ is closed.
- **(d)** $T(k).v$ is closed.

Idea of the proof

$G.v$ is closed $\Rightarrow$ $T.v$ is closed: By using the closed orbit lemma, let T act on $G.v$, there is a closed orbit $T.gv$. Hence, $(g^{-1}Tg).v$ is closed. Since G is nilpotent, $g^{-1}Tg = T$, thus $T.v$ is closed.

Remarks

By a well-known theorem of Mostow, any connected algebraic group G over a field k of char. 0 has a decomposition $G = L \cdot U$, where L is a maximal connected reductive k-subgroup. The groups which are direct product of a reductive group and an unipotent group are perhaps the best possible for the fact that "$G.v$ is closed if and only if $G(k).v$ is closed". The following example gives a minimum example for which $G.v$ is not Zariski closed although $G(k).v$ is closed.

Example

Assume further that k is a local field of characteristic 0, $G = \mathrm{SL}_2$, B the Borel subgroup of G, consisting of upper triangular matrices. Consider the standard representation of G by letting G act on the space $V_2 = \bar{k}[X, Y]_2$, considered as 3-dimensional $\bar{k}$-vector space with the canonical basis $\{X^2, XY, Y^2\}$. Then for $v = (1, 0, 1) \in V_2$, we have

(1) $B.v = \{(x, y, z) \mid 4xz = y^2 + 4\} \setminus \{z = 0\}$ is not Zariski closed;

(2) $B(k).v = \{(a^2 + b^2, 2bd, d^2) \mid ad = 1, a, b, c, d \in k\}$ is closed in the Hausdorff topology, where k is either $\mathbb{R}$ or a $\mathfrak{p}$-adic field, with $p = 2$ or $p \equiv 3 \pmod 4$.

Also, in the case of solvable groups, in contrast with the nilpotent case we have

Proposition

Let G be a smooth affine solvable algebraic group defined a local field k of characteristic 0, T an arbitrary maximal k-torus of G, and let G act k-regularly on an affine k-variety V, $v \in V(k)$. We consider the following statements.

(1) $G.v$ is closed in Zariski topology,

(2) For any above T, $T.v$ is closed in Zariski topology,

(3) $G(k).v$ is closed in Hausdorff topology,

(4) For any above T, $T(k).v$ is closed in Hausdorff topology.

Then we have the following logical scheme (2) $\Leftrightarrow$ (4), (1)$\Rightarrow$(3), (1) $\not\Rightarrow$(2), (2) $\not\Rightarrow$(1), (3) $\not\Rightarrow$(4), (4) $\not\Rightarrow$(3), (3) $\not\Rightarrow$(1).

On the topology on group cohomology of algebraic groups over complete valued fields

Set up

(1) Let G be an affine algebraic group scheme over $k \rightsquigarrow$ can define the flat cohomology sets (or groups) $\mathrm{H}^i_{fl}(k, G)$ for $i = 0, 1$. If G is commutative, we can define the group $\mathrm{H}^i_{fl}(k, G)$ for $i \geq 2$.

(2) If G is smooth (i.e., absolutely reduced ($k[G] \otimes_k \overline{k}$ is reduced)), then $\mathrm{H}^i_{fl}(\overline{k}/k, G) = \mathrm{H}^i(k_s/k, G)$.

(3) If k is endowed with a topology, say a v-adic topology, then $\mathrm{H}^0_{fl}(k, G) = G(k)$ has induced v-adic topology.

Set up (continued)

(4) In Shatz (1964, 1972), Milne (1986), due to the need of duality theory over local fields, a natural topology on the groups of cohomology has been introduced for commutative group scheme (only) and shows many applications: Tate-Nakayama duality, ...

(5) **Question:** Are the connecting maps continuous? What can one say in the non-commutative case?

(6) In Thang-Tan (2008), the special topology has been introduced. In this paper, we establish a relation between special topology and the canonical topology defined by S. Shatz.

Theorem (B.-N. Q. Thang)

Let G be an affine group scheme of finite type defined over a local field k. Then

(1) The (adelic) special and canonical topologies on $\mathrm{H}^1_{fl}(k, G)$ concide.

(2) Any connecting map appearing in the exact sequence of cohomology in degree ≤ 1 induced from a short exact sequence of a finite group schemes of finite type involving G is continuous w.r.t (adelic) canonical (or special) topologies.

Applications

Second proof for the Proposition 1.

Since G_v is smooth, by the above theorem the special (or canonical) topology on $\mathrm{H}^1_{fl}(k, G_v)$ is discrete. So from the exact sequence

$$G(k) \to (G.v)(k) \xrightarrow{\delta} \mathrm{H}^1_{fl}(k, G_v),$$

$G(k).v = \delta^{-1}(1)$ is open and closed in $(G.v)(k)$. Since δ is continuous, the conclusion holds for any other $G(k)$-orbit.

Special topology

(1) A smooth (i.e. linear) algebraic k-group H is called special (over k) (after Grothendieck-Serre) if

$$\mathrm{H}^1_{fl}(L, H) = \mathrm{H}^1(L, H) = 0 \text{ for all } L/k.$$

(2) Given a k-embedding $G \hookrightarrow H$ (special group), we have the following exact sequence of cohomology

$$1 \to G(k) \to H(k) \to (H/G)(k) \xrightarrow{\delta} \mathrm{H}^1_{fl}(k, G) \to 0.$$

Here H/G is a quasi-projective scheme of finite type defined over k.

(3) Since δ is surjective, by using the natural (Hausdorff) topology on $(H/G)(k)$, induced from that of k, we may endow $\mathrm{H}^1_{fl}(k, G)$ with the strongest topology such that δ is continuous.

(4) **Definition.** We call the above topology to be the H-special topology on $\mathrm{H}^1_{fl}(k, G)$.

Theorem

Let k be a local field and G an affine k-group scheme of finite type. Then the special topology on $\mathrm{H}^1_{fl}(k, G)$ does not depend on the choice of the embedding into special groups and it depends only on k-isomorphism class of G.

Canonical topology. Non-commutative case

(1) Let G be a non-commutative k-group scheme of finite type. Let $k \subseteq L \subseteq \overline{k}$ a normal extention. Let

$$\theta_L : Z^1(L/k, G(L)) \subseteq G(L^{\otimes 2}) \to Z^1(L/k, G(L))/\sim$$

be the quotient map.

(2) *Definition*

(a) The topology on $Z^1(L/k, G(L))$ induced from that of $G(L \otimes_k L)$ is called L/k-canonical topology on $Z^1(L/k, G(L))$.

(b) $\mathrm{H}^1_{fl}(L/k, G(L)) = Z^1(L/k, G(L))/\sim$ with quotient topology w.r.t. $\theta_L : Z^1(L/k, G(L)) \to Z^1/\sim$ is called L/k-canonical topology. (Denoted by $\tau_{L/k,c}$.)

(c) If $L = \overline{k}$, it is called canonical topology on $Z^1(\overline{k}/k, G(\overline{k}))$ (resp., on $\mathrm{H}^1_{fl}(k, G)$). (Denoted by τ_c.)

(d) The topology on $\mathrm{H}^1_{fl}(L/k, G(L))$ which is induced from τ_c on $\mathrm{H}^1_{fl}(k, G)$ will be denoted by $\tau_{c,L/k}$.

Theorem

Let k be a local field. Then for any affine k-group scheme of finite type G and any special k-embedding $G \hookrightarrow H$, the H-special topology on $\mathrm{H}^1(k, G)$ concides with canonical topology there. In particular, the H-special topology does not depend on the choice of the embedding $G \hookrightarrow H$.

Ideal of the proof

(1) We have the following commutative diagram

$$\begin{tikzcd} & & G(L^{\otimes 2}) \\ H(L) \supseteq & D'_k \arrow[r, "\varphi'"] \arrow[d, "q'"'] & Z^1_{fl}(L/k, G(L)) \arrow[u, "r"'] \arrow[d, "\theta_L"] \\ (H/G)(L) \supseteq (H/G)(k) \supseteq & D_k \arrow[r, "\varphi"] & \mathrm{H}^1_{fl}(L/k, G(L)) \subseteq \mathrm{H}^1_{fl}(k, G) \end{tikzcd}$$

where $D_k = \delta_k^{-1}(f_L(\mathrm{H}^1_{fl}(L/k, G(L))))$,
$D'_k = \{h \in H(L) \mid d_{H,1}(h) \in Z^1(L/k, G(L)) \subseteq G(L^{\otimes 2})\}$,
$q' = \pi_L|_{D'_k}$.

(2) Compare the quotient topology w.r.t the map φ $(D_k \subseteq (H/G)(k))$, is called $H - L/k$-special topology (denoted by τ_φ), and the topology which is induced from the special topology on $\mathrm{H}^1_{fl}(k, G)$.

Theorem

Let k be a local field, and let be given an exact sequence of affine k-group schemes of finite type:

$$1 \to A \to B \xrightarrow{\pi} C \to 1.$$

(1) All connecting maps between cohomology sets in degree ≤ 1 induced from the above diagram are continuous in their natural (i.e. v-adic or adelic) and the (adelic) special (resp. the canonical) topology on these sets.

(2) If A is central in B and C is smooth, then the coboundary map $\mathrm{H}^1_{fl}(k, C) \to \mathrm{H}^2_{fl}(k, A)$ is also continuous w.r.t. (adelic) canonical topologies.

(3) If, moreover, B is commutative, then all connecting maps in the exact sequence of flat cohomology, induced from above exact sequence, are continuous w.r.t. (adelic) canonical topology.

Proposition

If k as above, $X \hookrightarrow Y$ (as a closed subgroup scheme) Y/X is smooth, then the induced map $\mathrm{H}^1_{fl}(k, X) \to \mathrm{H}^1_{fl}(k, Y)$ is open in the special (thus also in canonical) topology on $\mathrm{H}^1_{fl}(k, X)$ and $\mathrm{H}^1_{fl}(k, Y)$.

Corollary

If k is as above, then for any smooth k-group G, the canonical (or special) topology on $\mathrm{H}^1(k, G)$ is the discrete topology.

Remark

The above Corollary is not valid anymore if we drop the smoothness assumption. Indeed, $G = \alpha_p$ implies that $\mathrm{H}^1_{fl}(k, G) = k/k^p$, and that k^p is closed but not open in k. Thus the trivial class $\{*\}$ is merely closed and not open in the special topology.

Some estimes for actions of algebraic tori over complete fields

Some questions

Let k be a global field (a finite extension of $\mathbb{Q}$ or a finite separable extension of $\mathbb{F}_p(T)$), T a k-torus, and let v be a discrete valuation on k, k_v the completion of k with respect to v, $\mathcal{O}_v$ the ring of v-adic integers.

(1) **(Bruhat-Tits' question)** Let $T(\mathcal{O}_v)$ be the maximal compact subgroup of $T(k_v)$. Then $T(k)T(\mathcal{O}_v) = T(k_v)$?

(2) **(Weak approximation question)** Let S be a finite set of inequivalent valuations over k and $\varphi : T(k) \to \prod_{v\in S} T(k_v)$ the diagonal embedding. Then φ has dense image in $\prod_{v\in S} T(k_v)$?

(3) **(Rationality question)** Is the variety T always k-rational?

Remark

Although there are some norm tori providing counter-examples for weak approximation question and rationality question, Colliot-Thelene/Sansuc (1987) showed that Bruhat-Tits question is valid for all norm tori. We propose the following question related to the Bruhat-Tits' question.

Question

Assume that T acts k-morphically on an affine k-variety X. Estimate the large measure of $T(k)X(\mathcal{O}_v)$ in $X(k_v)$ and study some topological properties of $T(k)X(\mathcal{O}_v)$.

Proposition (split torus of dim 1)

Let $\rho\colon G = \mathbb{G}_m \to \mathrm{GL}(V)$ be a representation defined over k, $k = \mathbb{Q}$, $v = v_p$. Then

(1) If $V = V_0$ then $G(k)V(\mathcal{O}_v) = V(\mathcal{O}_v)$. This set is closed, open, and compact in $V(k_v)$.

(2) We denote $I := \{i \in \mathbb{Z} \mid \text{the character}\, \alpha \mapsto \alpha^i \text{ is a weight of}\, \rho\}$. If all of elements of I are positive (or negative) integers, then we have $G(k)V(\mathcal{O}_v) = V(k_v)$.

(3) In all remaining cases of I, $G(k)V(\mathcal{O}_v)$ contains a subset Ω such that Ω is closed, open, non-compact and $G(k)V(\mathcal{O}_v) \subsetneqq V(k_v)$. The set $G(k)V(\mathcal{O}_v)$ is open and also non-compact in $V(k_v)$.

Theorem (B.)

Let $k = \mathbb{Q}$, $G = G_d = \left\{ \begin{pmatrix} a & b \\ db & a \end{pmatrix} \middle| \, a^2 - db^2 = 1 \right\}$ be the norm torus, $d \in \mathbb{Z}$, V a $\overline{k}$-vector space of dimension n, $\rho : G \to GL(V)$ a representation defined over k, $d = p^{2r}.d'$, $r \in \mathbb{N}$, $v_p(d') \in \{0, 1\}$.

(1) $v_p(d') = 0$.

(a) If $p = 2, d \equiv 3 \pmod 4$ then there exists $m \in \mathbb{Z}$, $m \leq 0$ such that
$V(\mathcal{O}_v) = \mathcal{O}_v \times \cdots \times \mathcal{O}_v \subseteq G(k)V(\mathcal{O}_v) \subseteq p^m(\mathcal{O}_v \times \cdots \times \mathcal{O}_v)$.

Theorem (B., continued)

(1) **(b)** If $p = 2$, $d \equiv 1 \pmod 4$ then $\overline{G(k)V(\mathcal{O}_v)}$ is compact if and only if $sup\{v_p(a^2 - db^2) \mid a, b \in \mathbb{Z}\} < +\infty$.

(c) If $p \neq 2$, $\left(\dfrac{d}{p}\right) = -1$, then there exists $m \in \mathbb{Z}$, $m \leq 0$ such that $V(\mathcal{O}_v) = \mathcal{O}_v \times \cdots \times \mathcal{O}_v \subseteq G(k)V(\mathcal{O}_v) \subseteq p^m(\mathcal{O}_v \times \cdots \times \mathcal{O}_v)$.

Theorem (B., continued)

(1) **(d)** If $p \neq 2$, $\left(\frac{d}{p}\right) = 1$, then there exist $r, r' \in \mathbb{Z}$, a basis $(v_1, \omega_1, \ldots, v_q, \omega_q, v'_1, \ldots, v'_l)$ of $V(\mathbb{Q})$, $n_1, \ldots, n_q \in \mathbb{Z}$, such that for
$A = \{x_1 v_1 + y_1\omega_1 + \cdots + x_q v_q + y_q\omega_q + x'_1 v'_1 + \cdots + x'_l v'_l \mid x_i^2 - \frac{y_i^2}{d} \in \mathcal{O}_v, \forall i = \overline{1, q}, x'_1, \ldots, x'_l \in \mathcal{O}_v\}$,
$B = \{x_1 v_1 + y_1\omega_1 + \cdots + x_q v_q + y_q\omega_q + x'_1 v'_1 + \cdots + x'_l v'_l \mid v_p(x_i^2 - \frac{y_i^2}{d}) \geq |n_i|, \forall\, i = \overline{1, q}, x'_1, \ldots, x'_l \in \mathcal{O}_v\}$, we have $p^r B \subseteq G(k_v)X(\mathcal{O}_v) \subseteq p^{r'} A$ and $G(k)X(\mathcal{O}_v)$ are dense in $G(k_v)X(\mathcal{O}_v)$.

(2) If $v_p(d') = 1$ then $V(\mathcal{O}_v) = \mathcal{O}_v \times \cdots \times \mathcal{O}_v \subseteq G(k)V(\mathcal{O}_v) \subseteq p^m V(\mathcal{O}_v)$, for some integer m.

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