In each of the following pairs of problems, we first find a particular solution using the method of variation of parameters using the given fundamental set of solutions $\{x_1, x_2\}$, and then we find a solution of the specified initial value problem.

For the method of variation of parameters we use Theorem 4.7.1 in the special case that $P(t) = A$ is a constant matrix. If $X(t)$ is a fundamental matrix for the homogeneous equation $x' = Ax$, then

$$x_p(t) = X(t) \int X^{-1}(t) g(t)\, dt$$

is a particular solution to the inhomogeneous equation $x' = Ax + g(t)$.

Note that $X^{-1}(t) = \frac{1}{W(t)} \operatorname{adj}(X(t))$, where $\operatorname{adj}(X(t))$ is the adjugate matrix of $X(t)$, and $W(t) = \det X(t)$ is the Wronskian.

To solve the initial value problem once we have $x_p(t)$ we solve for c_1 and c_2 so that $x(0) = c_1 x_1(0) + c_2 x_2(0) + x_p(0)$ equals the prescribed initial value.

Problems 2 and 6

$$x' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 4t \end{pmatrix}, \; x_1 = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; x_2 = e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

The Wronskian is $W(t) = \det X(t) = -2e^{-6t}$, so we have

$$\begin{aligned} u(t) &= \int X^{-1}(t) g(t)\, dt = -\frac{1}{2} \int e^{6t} \begin{pmatrix} -e^{-4t} & -e^{-4t} \\ -e^{-2t} & e^{-2t} \end{pmatrix} \begin{pmatrix} 1 \\ 4t \end{pmatrix} dt \\ &= \frac{1}{2} \int \begin{pmatrix} e^{2t}(1+4t) \\ e^{4t}(1-4t) \end{pmatrix} dt = \frac{1}{4} \begin{pmatrix} e^{2t}(-1+4t) \\ e^{4t}(\;\;1-2t) \end{pmatrix}. \end{aligned}$$

Thus, our particular solution will be

$$x_p(t) = X(t) u(t) = \frac{1}{4} \begin{pmatrix} -1+4t+1-2t \\ -1+4t-1+2t \end{pmatrix} = \frac{1}{2} \begin{pmatrix} t \\ -1+3t \end{pmatrix}.$$

To find the solution of the initial value problem, we need to find c_1 and c_2 so that

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

We must have $c_1 = 3/4$ and $c_2 = 5/4$, so our solution is

$$x(t) = \frac{3}{4} x_1(t) + \frac{5}{4} x_2(t) + x_p(t) = \frac{1}{4} \begin{pmatrix} 3e^{-2t} + 5e^{-4t} + 2t \\ 3e^{-2t} - 5e^{-4t} - 2 + 6t \end{pmatrix}.$$

Problems 3 and 7

$$x' = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} e^{-t} \\ t \end{pmatrix},\ x_1 = e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix},\ x_2 = e^{t}\begin{pmatrix} -1 \\ 1 \end{pmatrix},\ x(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The Wronskian is $W(t) = \det X(t) = 1$, so we have

$$\begin{aligned} u(t) &= \int X^{-1}(t)g(t)\,dt = \int \begin{pmatrix} e^t & e^t \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} e^{-t} \\ t \end{pmatrix} dt \\ &= \int \begin{pmatrix} 1 + te^t \\ te^{-t} \end{pmatrix} dt = \begin{pmatrix} t + te^t - e^t \\ -te^{-t} - e^{-t} \end{pmatrix}. \end{aligned}$$

Thus, our particular solution will be

$$\begin{aligned} x_p(t) = X(t)u(t) &= \begin{pmatrix} (t + te^t - e^t)e^{-t} + (-te^{-t} - e^{-t})(-e^t) \\ (-te^{-t} - e^{-t})e^t \end{pmatrix} \\ &= \begin{pmatrix} te^{-t} + t - 1 + t + 1 \\ -t - 1 \end{pmatrix} = \begin{pmatrix} te^{-t} + 2t \\ -t - 1 \end{pmatrix}. \end{aligned}$$

To find the solution of the initial value problem, we need to find c_1 and c_2 so that

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

We must have $c_1 = 1$ and $c_2 = 2$, so our solution is

$$x(t) = x_1(t) + 2x_2(t) + x_p(t) = \begin{pmatrix} e^{-t} - 2e^t + te^{-t} + 2t \\ 2e^t - t - 1 \end{pmatrix}.$$

Problems 4 and 8

$$x' = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} x + \begin{pmatrix} -1 \\ t \end{pmatrix},\ x_1 = e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix},\ x_2 = e^{-t}\begin{pmatrix} -1 \\ t \end{pmatrix},\ x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The Wronskian is $W(t) = \det X(t) = e^{-2t}$, so we have

$$\begin{aligned} u(t) &= \int X^{-1}(t)g(t)\,dt = \int e^{2t}e^{-t} \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ t \end{pmatrix} dt \\ &= \int \begin{pmatrix} 0 \\ e^t \end{pmatrix} dt = \begin{pmatrix} 0 \\ e^t \end{pmatrix}. \end{aligned}$$

Thus, our particular solution will be

$$x_p(t) = X(t)u(t) = \begin{pmatrix} -1 \\ t \end{pmatrix}.$$

To find the solution of the initial value problem, we need to find c_1 and c_2 so that

$$c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We must have $c_1 = 0$ and $c_2 = -2$, so our solution is

$$x(t) = -2x_2(t) + x_p(t) = \begin{pmatrix} 2e^{-t} - 1 \\ -2te^{-t} + t \end{pmatrix}.$$

Problems 5 and 9

$$x' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \; x_1 = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \; x_2 = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \; x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The Wronskian is $W(t) = \det X(t) = 1$, so we have

$$\begin{aligned} u(t) &= \int X^{-1}(t) g(t)\, dt = \int \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} dt \\ &= \int \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt = \begin{pmatrix} t \\ 0 \end{pmatrix}. \end{aligned}$$

Thus, our particular solution will be

$$x_p(t) = X(t)u(t) = \begin{pmatrix} t\cos t \\ -t\sin t \end{pmatrix}.$$

To find the solution of the initial value problem, we need to find c_1 and c_2 so that

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We must have $c_1 = c_2 = 1$, so our solution is

$$x(t) = x_1(t) + x_2(t) + x_p(t) = \begin{pmatrix} \cos t + \sin t + t\cos t \\ -\sin t + \cos t - t\sin t \end{pmatrix}.$$