Problem Set 4

3.4.1

$$\vec{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \vec{x}.$$

Sol:

Let $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$.

We have $\det(\lambda I - A) = \lambda^2 - 2\lambda + 5$.

$\lambda_{1,2} = 1 \pm 2i$.

For eigenvalue $\lambda_1 = 1 + 2i$, we have eigenvector $\vec{V}_1 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$.

Then we have a complex solution

$$\vec{y}(t) = e^{\lambda_1 t} \vec{V}_1 = e^{(1+2i)} \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

$$= e^t \begin{pmatrix} \cos(2t) - \sin(2t) \\ 2\cos(2t) \end{pmatrix} + i e^t \begin{pmatrix} \sin(2t) + \cos(2t) \\ 2\sin(2t) \end{pmatrix}$$

Then we have two real solutions

$$\vec{x}_1(t) = e^t \begin{pmatrix} \cos(2t) - \sin(2t) \\ 2\cos(2t) \end{pmatrix}, \quad \vec{x}_2(t) = e^t \begin{pmatrix} \sin(2t) + \cos(2t) \\ 2\sin(2t) \end{pmatrix}$$

with Wronskian

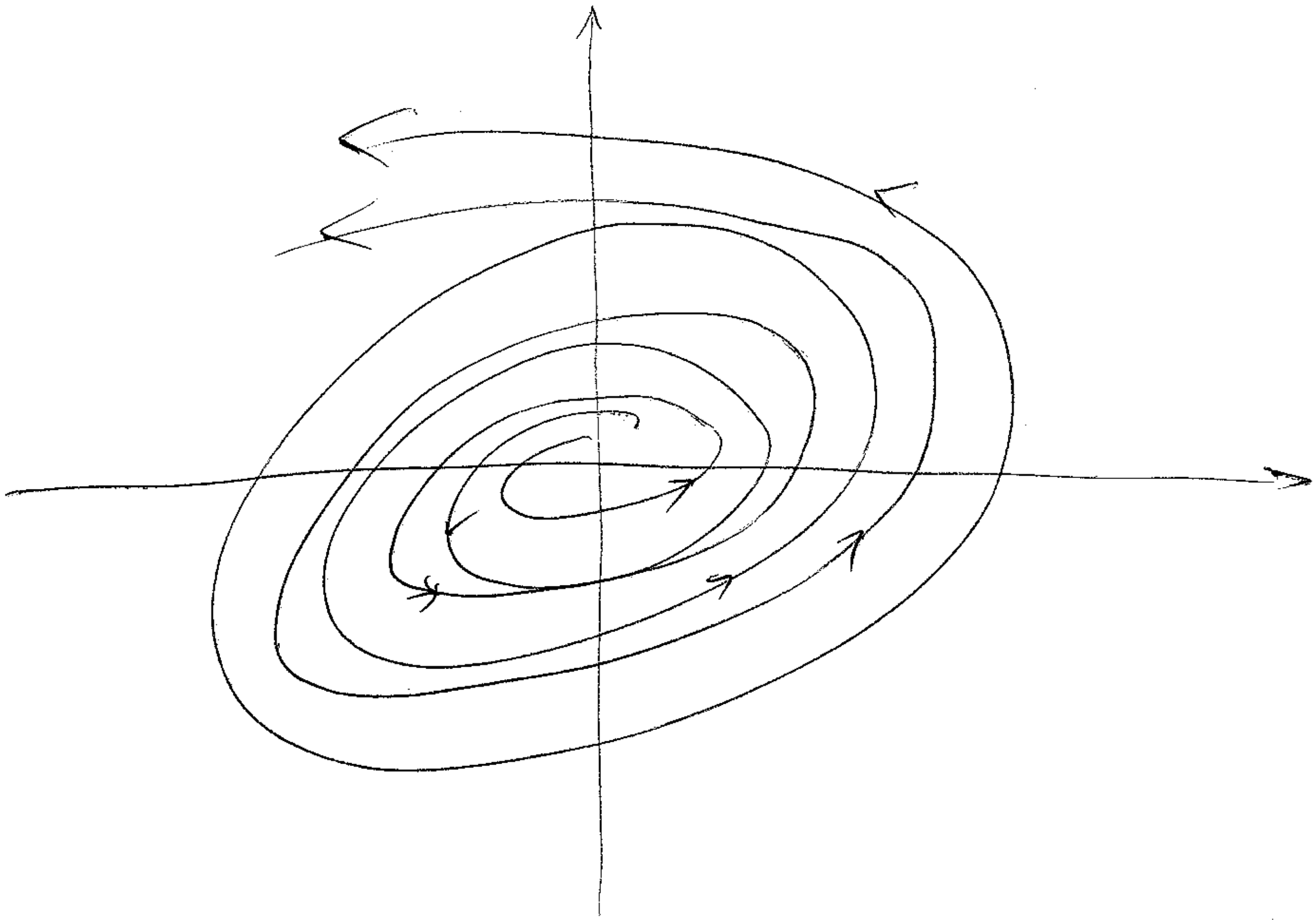
$$W[\vec{x}_1(t), \vec{x}_2(t)] \neq 0.$$

Therefore $\{\vec{x}_1(t), \vec{x}_2(t)\}$ is a fundamental set of solutions, and the general solution is

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t).$$

Obviously, the origin $\vec{0}$ is the only equilibrium solution. Because $Re(\lambda_1) = 1 > 0$, we know the equilibrium solution is unstable. Except for the equilibrium solution, all other solutions spiral outward and the trajectories are unbounded as $t \to \infty$

Sketch of phase portrait:



3.4.3 $\quad \vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x}$

Sol: $\det(\lambda I - A) = \lambda^2 + 1$

Eigenvalues : $\lambda = \pm i$.

It's easy to calculate that $\lambda_1 = i$ has an eigenvector $\vec{V}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$.

Therefore, we have a complex solution

$$\vec{y}(t) = e^{\lambda_1 t} \vec{V}_1 = \begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} + i \begin{pmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{pmatrix}$$

Real solutions $\vec{x}_1(t) = \begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix}$, $\vec{x}_2(t) = \begin{pmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{pmatrix}$

have nonzero Wronskian, so they consist of a fundamental set of solutions.

The general solution is

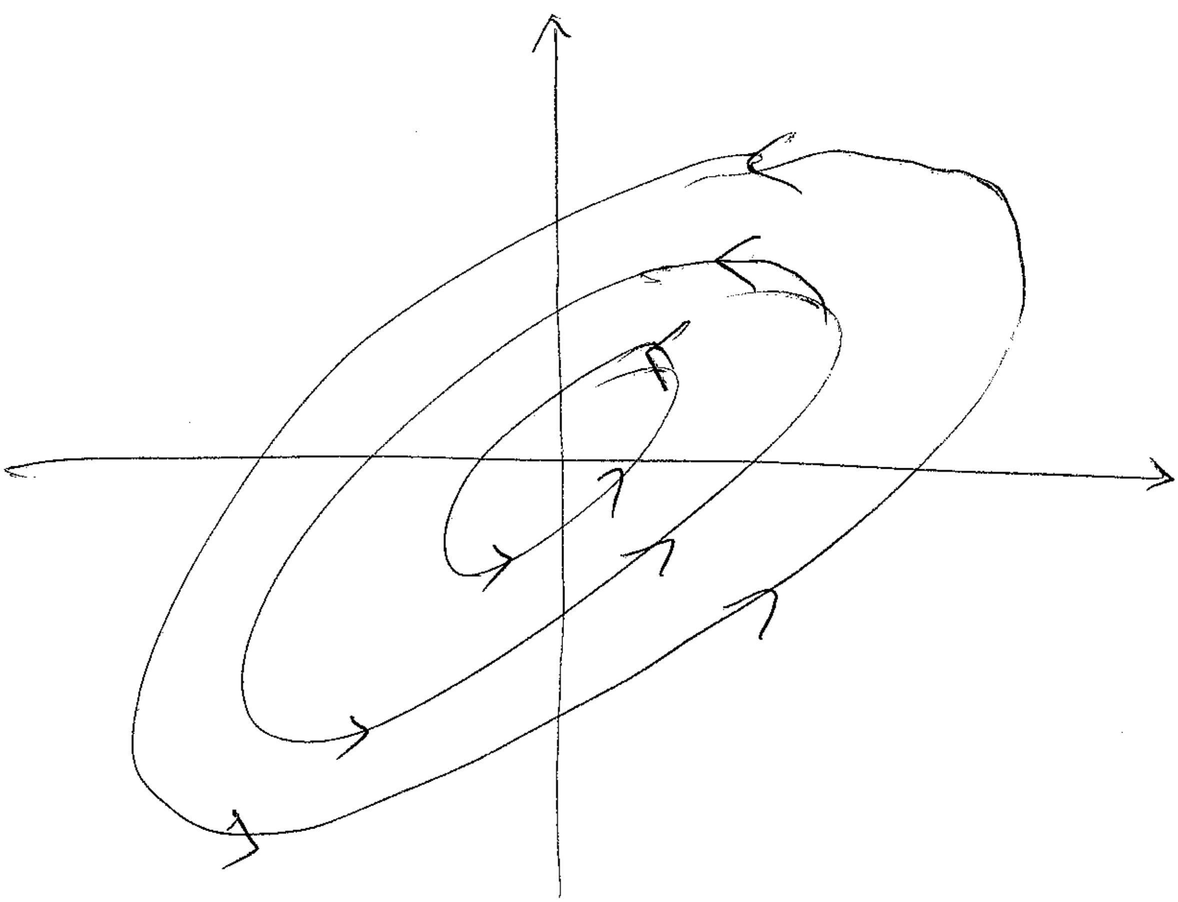
$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t).$$

The origin is the only equilibrium solution.

Because $Re(\lambda_1) = 0$, we know the equilibrium is a center which is stable but not asymptotically stable

Other solutions spiral ~~inward~~ in elipses and the trajectories are bounded as $t \to \infty$

Sketch of phase portrait:



3.4.7 $\quad \vec{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

Sol:

Let $A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$. We have $\det(I - \lambda A) = \lambda^2 + 2\lambda + 5$.

The the two eigenvalues are $\lambda_{1,2} = -1 \pm 2i$.

For eigenvalue $\lambda_1 = -1 + 2i$, we have eigenvector

$$\vec{v}_1 = \begin{pmatrix} 2i \\ 1 \end{pmatrix}.$$

Then ~~the~~ we have an complex solution

$$\vec{y}(t) = e^{\lambda_1 t} \vec{v}_1 = e^{-t} \begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} + i e^{-t} \begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix}.$$

Notice that the two real solutions

$$\vec{x}_1(t) = e^{-t} \begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix}, \quad \vec{x}_2(t) = e^{-t} \begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix}$$

have nonzero Wronskian, so they are a fundamental set of solutions.

Therefore, the general solution is

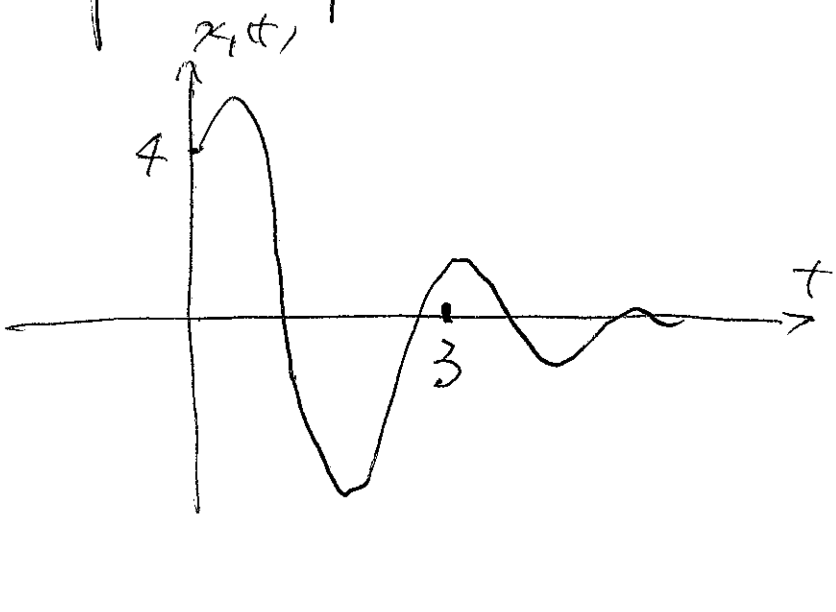
$$\vec{x}(t) = C_1 e^{-t}\begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix}.$$

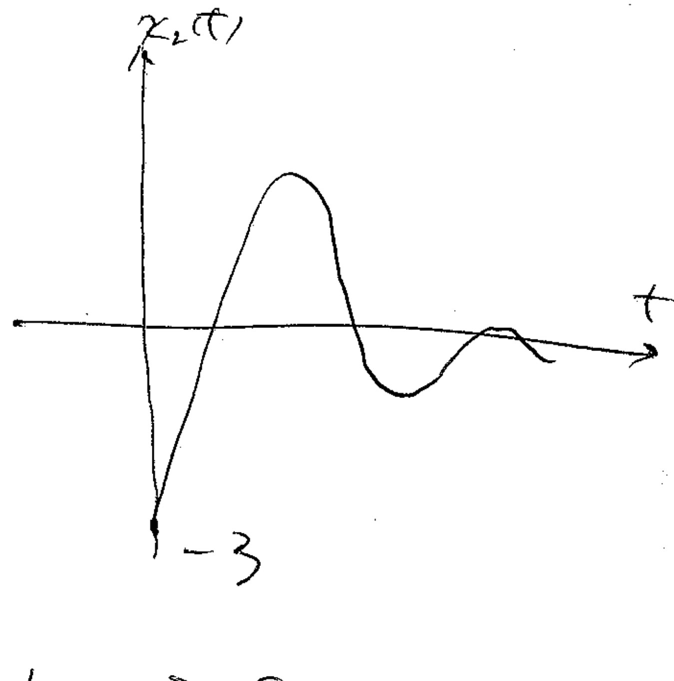
By using the initial condition $\vec{x}(0) = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$, we have $C_1 = -3$ and $C_2 = 2$.

Therefore the solution to the initial condition is

$$\vec{x}(t) = e^{-t}\begin{pmatrix} 4\cos(2t) + 6\sin(2t) \\ -3\cos(2t) + 2\sin(2t) \end{pmatrix}$$

Component plots:



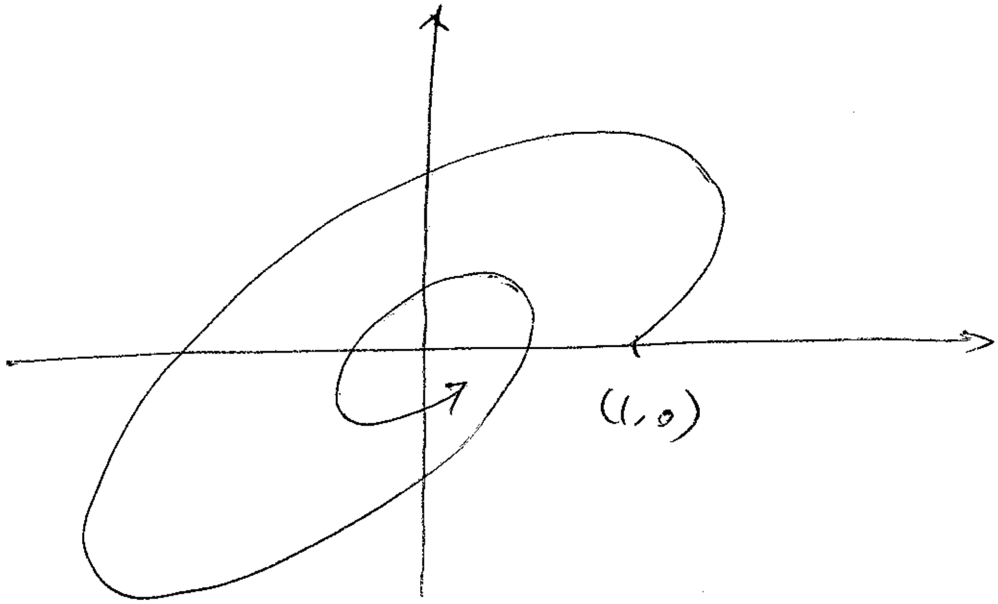


As $t \to \infty$, $x_1(t),\ x_2(t) \to 0$

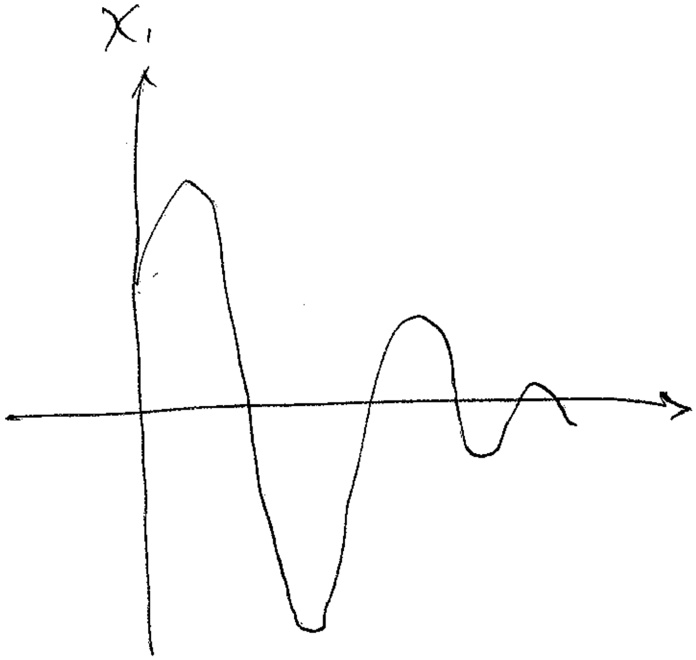
3.4.11.
(a) $\vec{x}' = \begin{pmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{pmatrix} \vec{x}$.

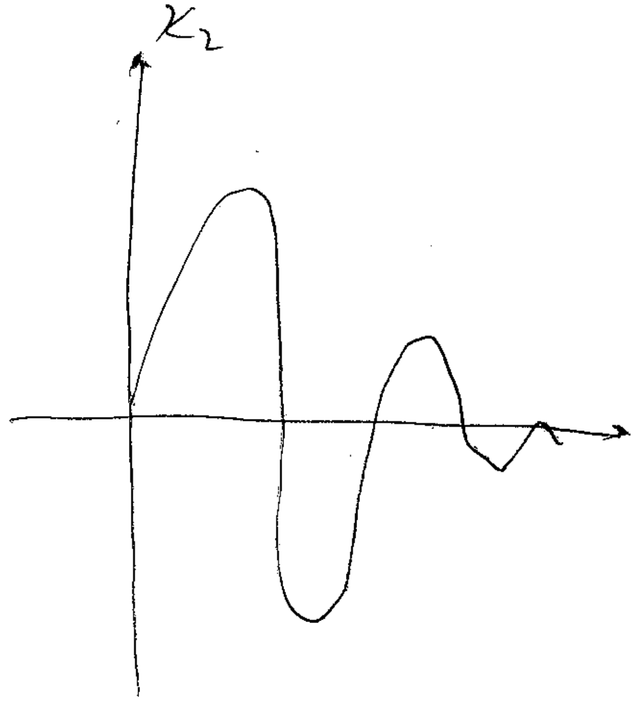
$\det(\lambda I - A) = 0 \Longrightarrow \lambda_{1,2} = -\frac{1}{4} \pm i$.

(b).



(c)





3.4.13.

(a) $A = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix}$ $\det(\lambda I - A) = 0$

$\Rightarrow \lambda_{1,2} = \alpha \pm i$.

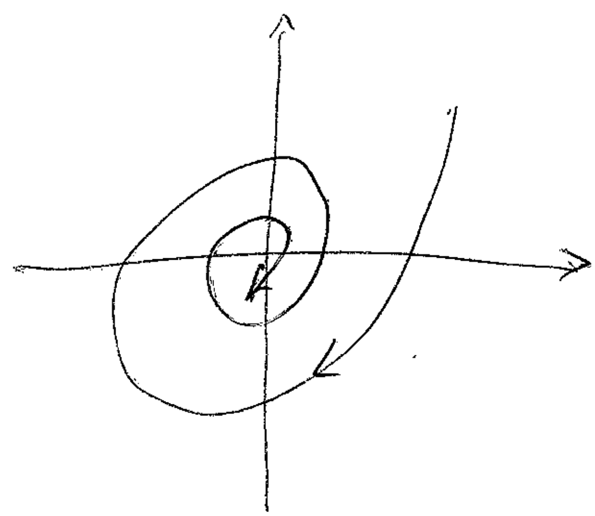
(b) $\alpha = 0$ is critical

$\alpha < 0$: trajectories sprial inward

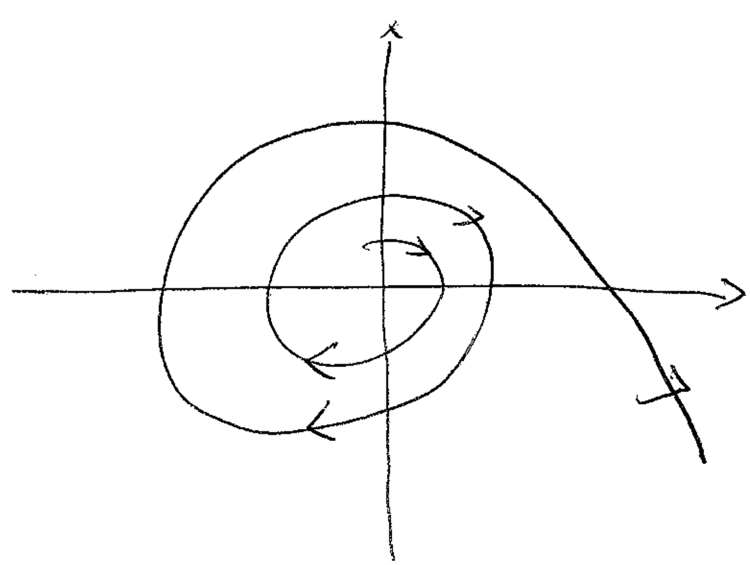
$\alpha > 0$: trajectories sprial outward.

(c) Phase portrait :

$\alpha < 0$



$\alpha > 0$



3.4.15

(a) $A = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix}$ $\det(\lambda I - A) = 0$

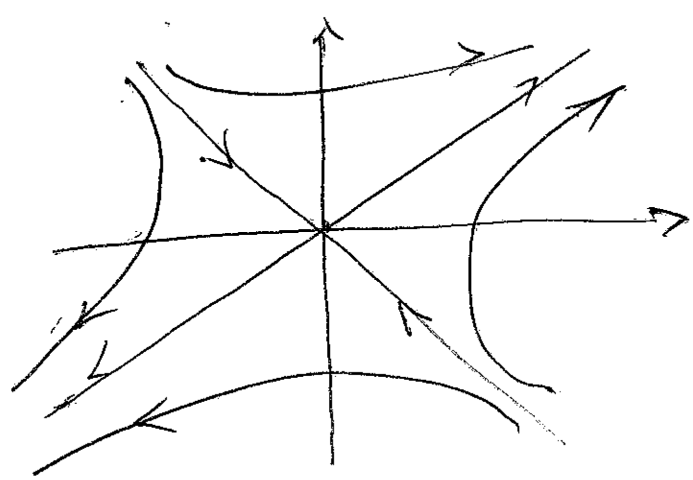
$\Rightarrow \lambda_{1,2} = \pm\sqrt{4-5\alpha}$

(b) $\alpha = \frac{4}{5}$ is critical

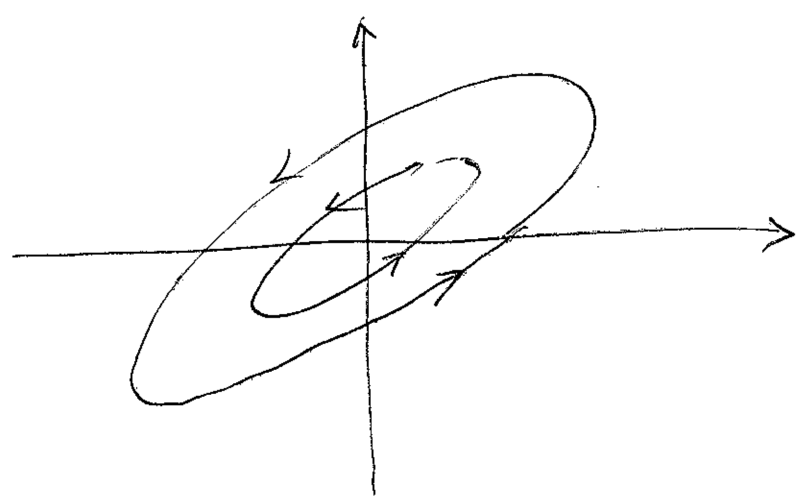
$\alpha < \frac{4}{5}$: the equilibrium solution is a ~~...~~ saddle point

$\alpha > \frac{4}{5}$: the equilibrium solution is a center

(c) $\alpha < \frac{4}{5}$:



$\alpha > \frac{4}{5}$:



3.4.20

(a) $A = \begin{pmatrix} 4 & \alpha \\ 8 & -6 \end{pmatrix}$. $\det(\lambda I - A) = 0$

$\Longrightarrow \quad \lambda_{1,2} = -1 \pm \sqrt{25 + 8\alpha}$

(b) Critical values $\alpha_1 = -\frac{25}{8}$, $\alpha_2 = -3$

$\epsilon > 0$ and small:

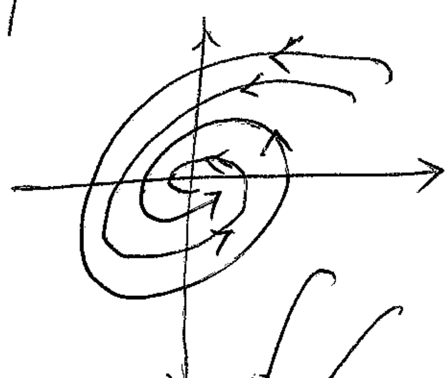
$\alpha = -\frac{25}{8} - \varepsilon$: the equilibrium is a spiral sink

$\alpha = -\frac{25}{8} + \varepsilon$: the equilibrium is a nodal sink

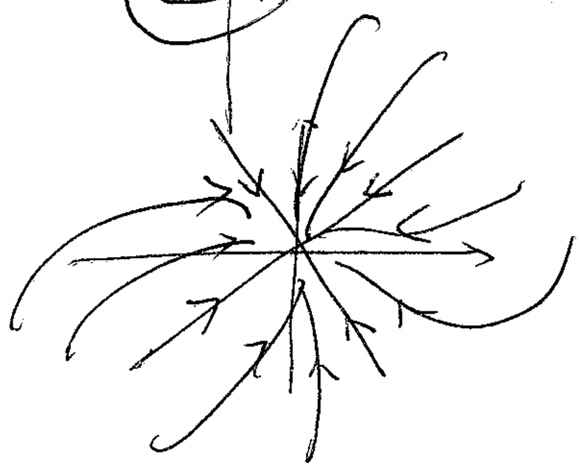
$\alpha = -3 - \varepsilon$: the equilibrium is a nodal sink

$\alpha = -3 + \varepsilon$: the equilibrium is a saddle point.

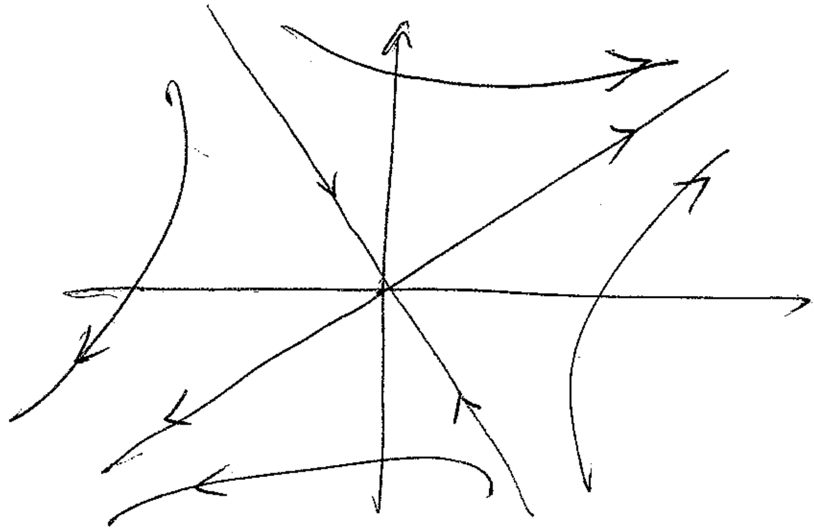
(c). $\alpha < -\frac{25}{8}$:



$-\frac{25}{8} < \alpha < -3$:



$\alpha > -3$



3.5.2

$$\vec{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{1}{4} \end{pmatrix} \vec{x}$$

Sol: $A = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{1}{4} \end{pmatrix}$ $\quad \det(\lambda I - A) = \left(\lambda - \frac{1}{2}\right)^2$.

Eigenvalues $\lambda_{1,2} = \frac{1}{2}$.

For eigenvalue $\lambda_1 = \frac{1}{2}$, we have eigenvector

$$\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Then

$$\vec{x}_1(t) = e^{\frac{t}{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Suppose another solution

$$\vec{x}_2(t) = t e^{\frac{t}{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{\frac{t}{2}} \vec{w}$$

$\vec{w}$ satisfies $\left(A - \frac{1}{2} I\right) \vec{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

we can choose $\vec{w} = \begin{pmatrix} 0 \\ \frac{4}{3} \end{pmatrix}$

Then $\vec{x}_2(t) = t e^{\frac{t}{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{\frac{t}{2}} \begin{pmatrix} 0 \\ \frac{4}{3} \end{pmatrix}$

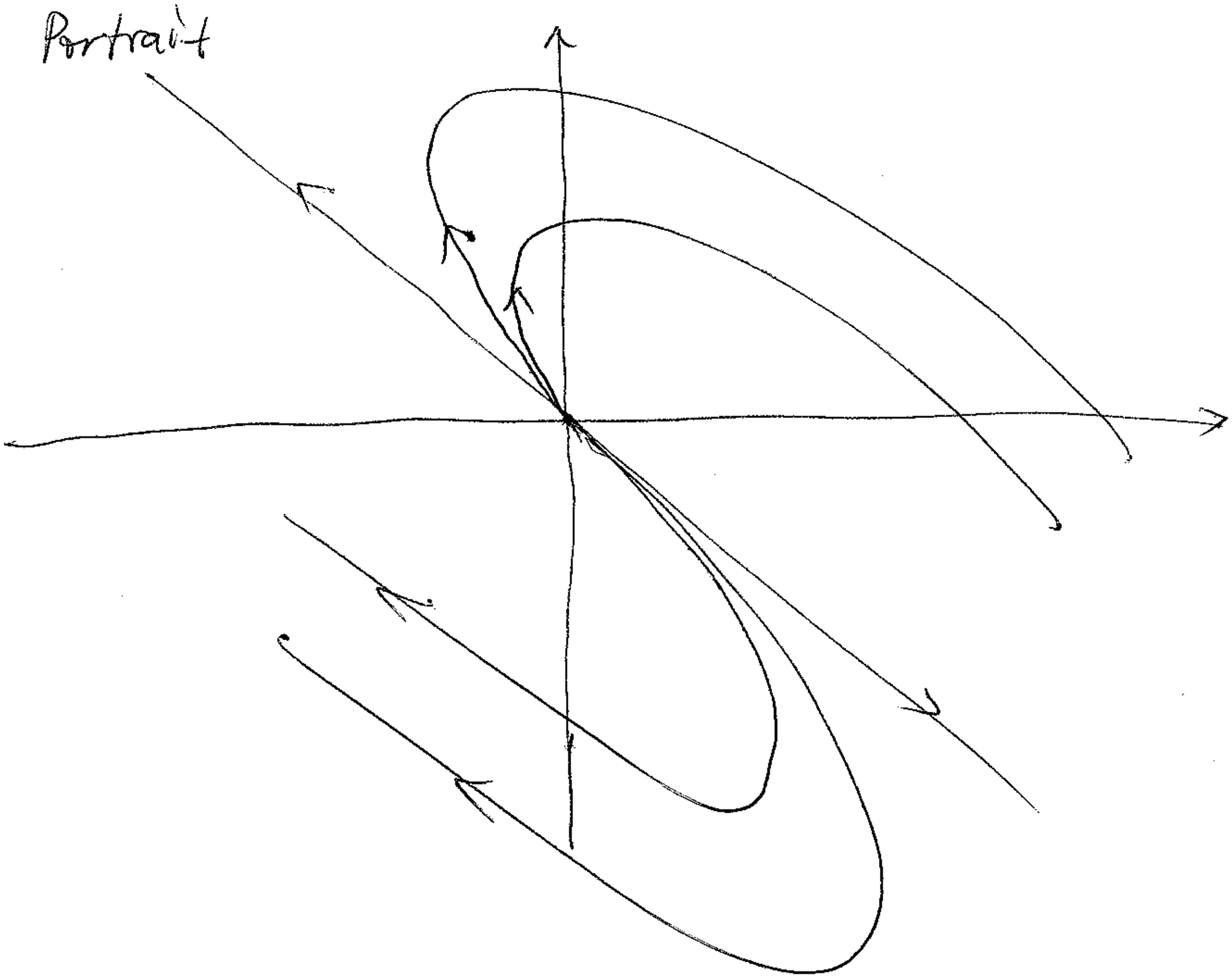
Then the general solution is

$$\vec{x}(t) = c_1 \vec{x_1}(t) + c_2 \vec{x_2}(t)$$

The equilibrium solution $\vec{0}$ is a nodal source.

All other trajectories will become unbounded as $t \to +\infty$

Phase Portrait



3.5.4

$$\vec{x}' = \begin{pmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{pmatrix} \vec{x}.$$

Sol: $A = \begin{pmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{pmatrix}$ $\quad \det(\lambda I - A) = \left(\lambda + \frac{1}{2}\right)^2$

Eigenvalues $\lambda_{1,2} = -\frac{1}{2}$.

Eigenvector $(A - \lambda I)\vec{v} = 0 \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$\vec{x}_1(t) = e^{-\frac{t}{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Suppose $\vec{x}_2(t) = t e^{-\frac{t}{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-\frac{t}{2}} \vec{w}$

$(A - \lambda I)\vec{w} = \vec{v}$. Choose $\vec{w} = \begin{pmatrix} -\frac{2}{5} \\ 0 \end{pmatrix}$

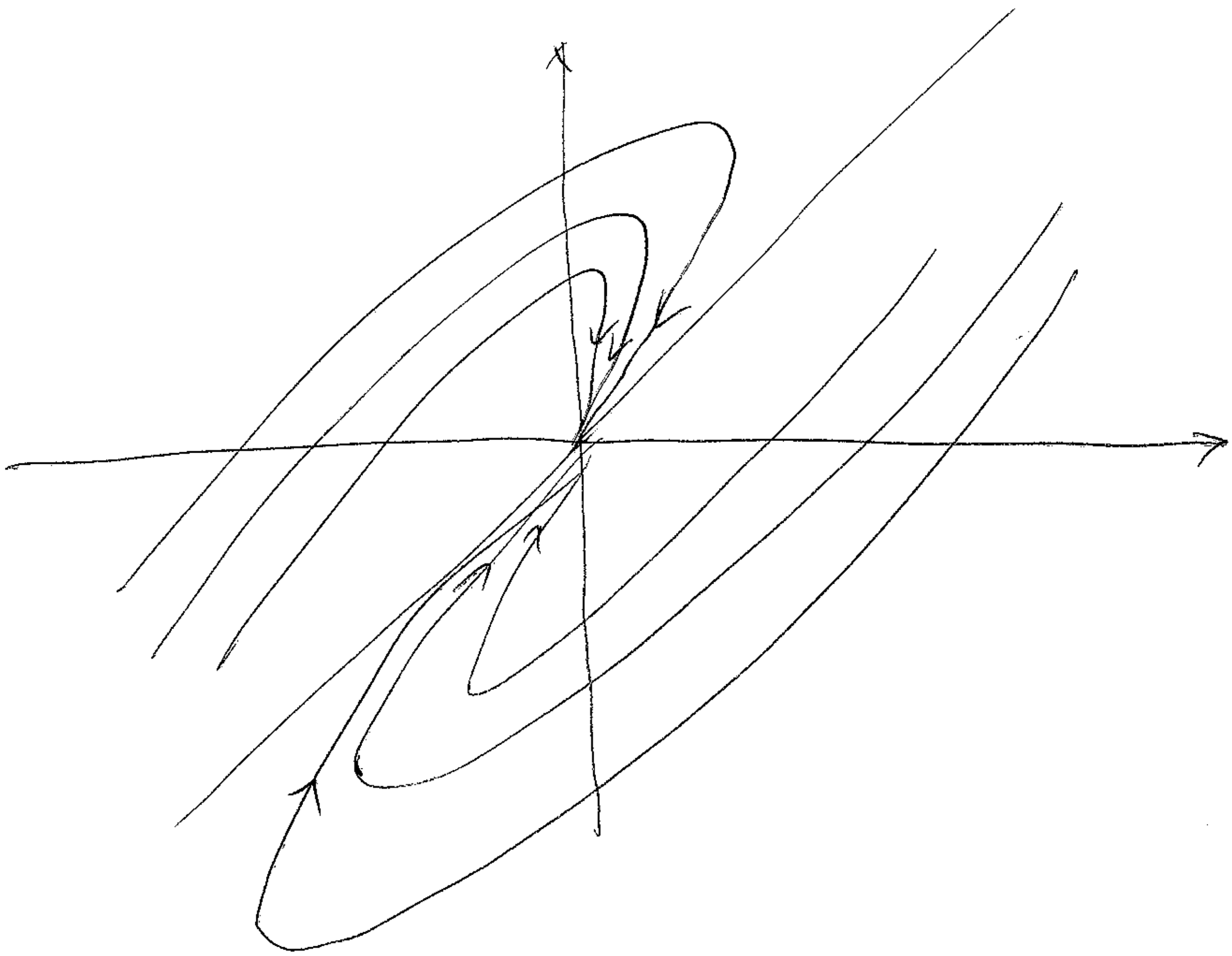
Then the general solution is

$$\vec{x}(t) = C_1 e^{-\frac{t}{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \left(t e^{-\frac{t}{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-\frac{t}{2}} \begin{pmatrix} -\frac{2}{5} \\ 0 \end{pmatrix} \right)$$

The equilibrium solution is a nodal source.

All other trajectories will approach to $\vec{0}$ as $t \to \infty$.

Phase portrait:



3.5.8. $\vec{x}' = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \vec{x}$, $\vec{x}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

Sol: Let $A = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$.

$\det(\lambda I - A) = (\lambda+1)^2 \Rightarrow$ Eigenvalues $\lambda_{1,2} = -1$.

Eigenvector $(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

First solution $\vec{x}_1(t) = e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Suppose another solution $\vec{x}_2(t) = te^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-t}\vec{w}$.

$(A - \lambda I)\vec{w} = \vec{v}$. We choose $\vec{w} = \begin{pmatrix} -\frac{2}{3} \\ 0 \end{pmatrix}$.

The general solution is

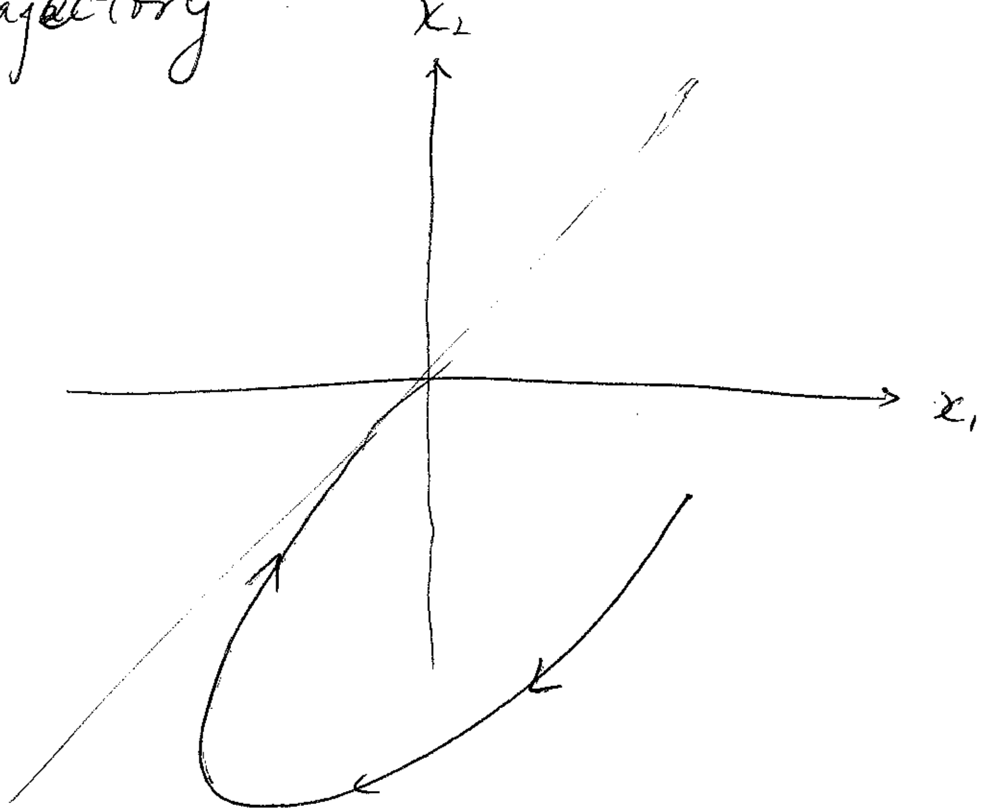
$$\vec{x}(t) = C_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\left(te^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-t}\begin{pmatrix} -\frac{2}{3} \\ 0 \end{pmatrix}\right).$$

By the initial condition $\vec{x}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, we have

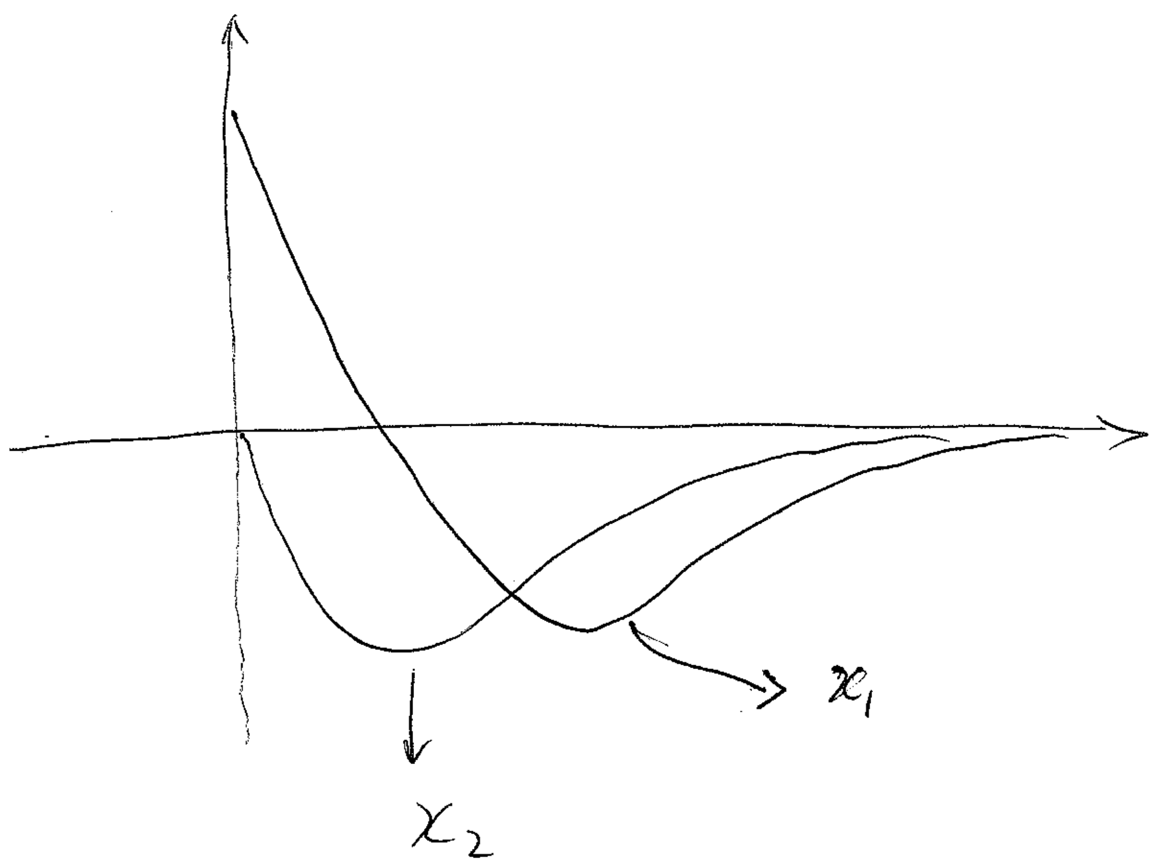
$C_1 = -1$, $C_2 = -6$, and

$$\vec{x}(t) = -e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 6\left(te^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-t}\begin{pmatrix} -\frac{2}{3} \\ 0 \end{pmatrix}\right)$$

Trajectory



Component plots



3.5.10 $\quad \vec{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{1}{4} \end{pmatrix} \vec{x} \qquad \vec{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$

Sol: From 3.5.2, we know the general solution

$$\vec{x}(t) = C_1 e^{\frac{t}{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \left(t e^{\frac{t}{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{\frac{t}{2}} \begin{pmatrix} 0 \\ \frac{4}{3} \end{pmatrix} \right)$$

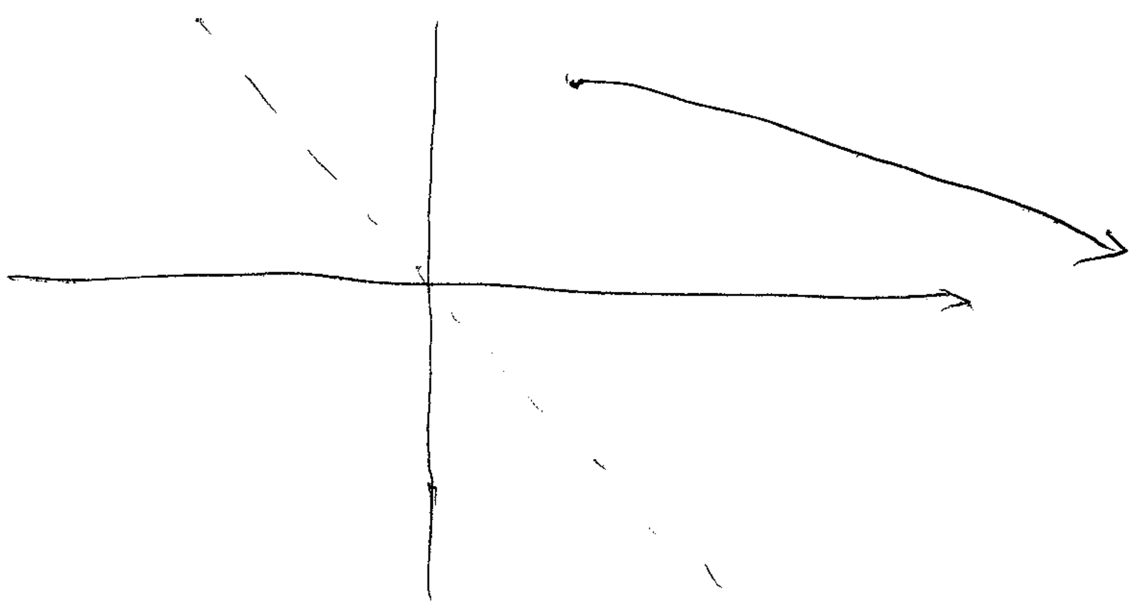
By the initial condition $\vec{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, we have

$$\begin{cases} C_1 = 2 \\ C_2 = \frac{15}{4}. \end{cases}$$

Then we have the solution to the initial value problem

$$\vec{x}(t) = e^{\frac{t}{2}} \begin{pmatrix} 2 + \frac{15}{4} t \\ 3 - \frac{15}{4} t \end{pmatrix}$$

Trajectory



Components plots :