MATH 53 HOMEWORK 4 SOLUTION

Section 3.1 problems: 13, 15, 20.
Section 3.2 problems: 2,3,8, 15, 18, 20.
Section 3.3 problems: 1, 3, 5,7, but don't draw the direction field or phase portrait – just find the solution and describe behavior as t goes infinity.

Problem 3.1.13

$$\lambda I - A = \begin{bmatrix} \lambda-3 & 2 \\ -2 & \lambda+2 \end{bmatrix} \quad\Rightarrow\quad (\lambda-3)(\lambda+2)+4=0 \quad\Rightarrow\quad \lambda^2-\lambda-2=0 \quad\Rightarrow\quad \lambda=2,-1$$

For $\lambda = 2$,

$$(2I-A)\vec{v}=0 \quad\Rightarrow\quad \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=0 \quad\Rightarrow\quad -v_1+2v_2=0 \quad\Rightarrow\quad \vec{v}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For $\lambda = -1$,

$$(-I-A)\vec{v}=0 \quad\Rightarrow\quad \begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=0 \quad\Rightarrow\quad -2v_1+v_2=0 \quad\Rightarrow\quad \vec{v}=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Problem 3.1.15

$$\lambda I - A = \begin{bmatrix} \lambda-3 & 4 \\ -1 & \lambda+1 \end{bmatrix} \quad\Rightarrow\quad (\lambda-3)(\lambda+1)+4=0 \quad\Rightarrow\quad \lambda^2-2\lambda+1=0 \quad\Rightarrow\quad \lambda=1$$

For $\lambda = 1$, we find an eigenvector,

$$(I-A)\vec{v}=0 \quad\Rightarrow\quad \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=0 \quad\Rightarrow\quad -v_1+2v_2=0 \quad\Rightarrow\quad \vec{v}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Problem 3.1.20

$$\lambda I - A = \begin{bmatrix} \lambda-2 & 1 \\ -3 & \lambda+2 \end{bmatrix} \quad\Rightarrow\quad (\lambda-2)(\lambda+2)+3=0 \quad\Rightarrow\quad \lambda^2-1=0 \quad\Rightarrow\quad \lambda=1,-1$$

For $\lambda = 1$,

$$(I-A)\vec{v}=0 \quad\Rightarrow\quad \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=0 \quad\Rightarrow\quad -v_1+v_2=0 \quad\Rightarrow\quad \vec{v}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = -1$,

$$(-I-A)\vec{v}=0 \quad\Rightarrow\quad \begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=0 \quad\Rightarrow\quad -3v_1+v_2=0 \quad\Rightarrow\quad \vec{v}=\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Problem 3.2.2 It's non autonomous and non homogeneous because of the term $\sin t, \cos t$. The matrix notation is

$$\vec{\mathbf{x}}'(t) = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\vec{\mathbf{x}}(t) + \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$$

Problem 3.2.3 It's non autonomous because of the term $2t$ in front of x. It is homogeneous. The matrix notation is

$$\vec{\mathbf{x}}'(t) = \begin{bmatrix} -2t & 1 \\ 3 & -1 \end{bmatrix}\vec{\mathbf{x}}(t)$$

Problem 3.2.8 It's both autonomous and homogeneous. The matrix notation is

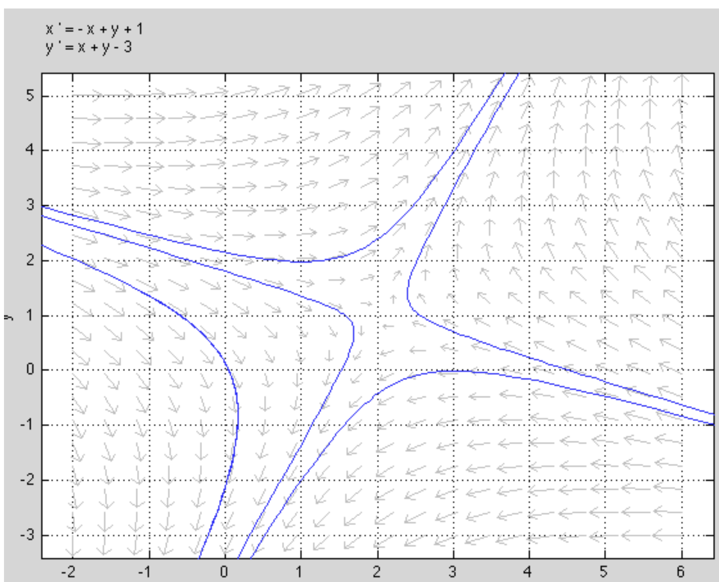
$$\vec{\mathbf{x}}'(t) = \begin{bmatrix} 3 & -4 \\ 1 & 3 \end{bmatrix} \vec{\mathbf{x}}(t)$$

Problem 3.2.15 (a) The equilibrium solution is the solution that makes $x' = y' = 0$. That is,

$$-x + y + 1 = 0, x + y - 3 = 0 \quad \Rightarrow \quad x = 2, y = 1$$

That is, $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the equilibrium solution.

(b)



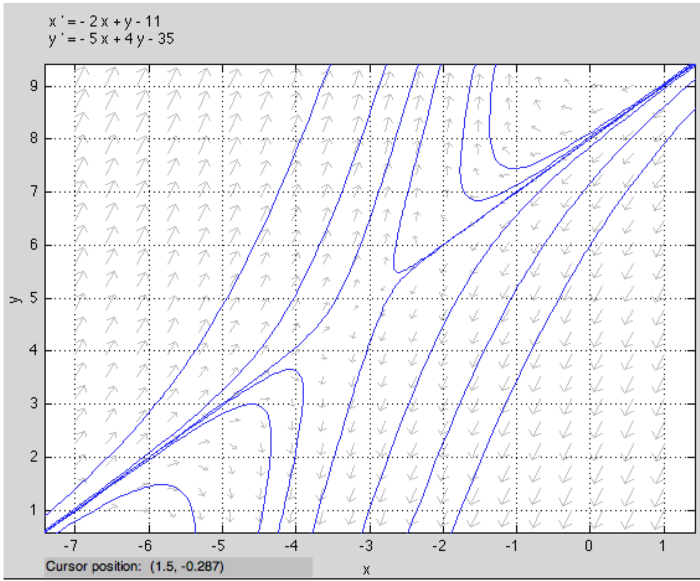
(c) From the plot, one see that the solutions of the system move away from the equilibrium points.

Problem 3.2.18 (a) Similarly, we solve

$$-2x + y - 11 = 0, -5x + 4y - 35 = 0 \quad \Rightarrow \quad x = -3, y = 5$$

That is, $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ is the equilibrium solution.

(b)



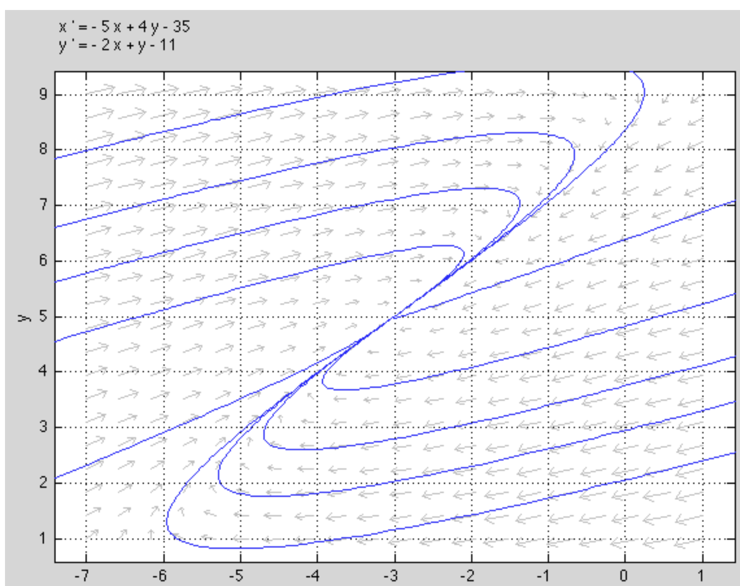
(c) From the plot, one see that most solutions of the system move away from the equilibrium points except those starting on one specific line. In that case, it will converge to (-3,5).

Problem 3.2.20 (a) Similarly, we solve

$$-5x + 4y - 35 = 0, -2x + y - 11 = 0 \quad \Rightarrow \quad x = -3, y = 5$$

That is, $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ is the equilibrium solution.

(b)



(c) From the plot, one see that all solutions of the system converges to (-3,5).

Problem 3.3.1 From problem 3.1.13, we have found eigenvalues and eigenvectors and get

$$\lambda_1 = 2, \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}; \quad \lambda_2 = -1, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Thus, we have found two solutions,

$$\vec{\mathbf{x}}_1(t) = e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{\mathbf{x}}_2(t) = e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and the general solution is given by

$$\vec{\mathbf{x}}(t) = C_1 e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

As $t \to \infty$, the second term goes to zero for any C_2. If $C_1 > 0$ then both $x(t), y(t)$ goes to plus infinity and approaches the line $y = x/2$. If $C_1 < 0$, then the solution goes to negative infinity and approaches the line $y = x/2$. If $C_1 = 0$, then the solution just approaches $(0, 0)$.

Problem 3.3.3 From problem 3.1.20, we have found eigenvalues and eigenvectors and get

$$\lambda_1 = 1, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \lambda_2 = -1, \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Thus, we have found two solutions,

$$\vec{\mathbf{x}}_1(t) = e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{\mathbf{x}}_2(t) = e^{-t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and the general solution is given by

$$\vec{\mathbf{x}}(t) = C_1 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

As $t \to \infty$, the second term goes to zero for any C_2. If $C_1 > 0$ then both $x(t), y(t)$ goes to plus infinity and approaches the line $y = x$. If $C_1 < 0$, then the solution goes to negative infinity and approaches the line $y = x$. If $C_1 = 0$, then the solution just approaches $(0, 0)$.

Problem 3.3.5 First we find eigenvalues and eigenvectors,

$$\lambda I - A = \begin{bmatrix} \lambda - 4 & 3 \\ -8 & \lambda + 6 \end{bmatrix} \quad \Rightarrow \quad (\lambda - 4)(\lambda + 6) + 24 = 0 \quad \Rightarrow \quad \lambda^2 + 2\lambda = 0 \quad \Rightarrow \quad \lambda = 0, -2$$

For $\lambda = 0$,

$$(0I - A)\vec{v} = 0 \quad \Rightarrow \quad \begin{bmatrix} -4 & 3 \\ -8 & 6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad \Rightarrow \quad -4v_1 + 3v_2 = 0 \quad \Rightarrow \quad \vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

For $\lambda = -2$,

$$(-2I - A)\vec{v} = 0 \quad \Rightarrow \quad \begin{bmatrix} -6 & 3 \\ -8 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad \Rightarrow \quad -6v_1 + 3v_2 = 0 \quad \Rightarrow \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Thus, we have found two solutions,

$$\vec{\mathbf{x}}_1(t) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \vec{\mathbf{x}}_2(t) = e^{-2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and the general solution is given by

$$\vec{\mathbf{x}}(t) = C_1\begin{bmatrix} 3 \\ 4 \end{bmatrix} + C_2 e^{-2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

As $t \to \infty$, the second term goes to zero for any C_2 and the solution simply converges to the point $(3C_1, 4C_1)$.

Problem 3.3.7 First we find eigenvalues and eigenvectors,

$$\lambda I - A = \begin{bmatrix} \lambda - \frac{5}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \lambda - \frac{5}{4} \end{bmatrix} \quad \Rightarrow \quad (\lambda - \frac{5}{4})^2 - (\frac{3}{4})^2 = 0 \quad \Rightarrow \quad \lambda - \frac{5}{4} = \pm\frac{3}{4} \quad \Rightarrow \quad \lambda = 2, \frac{1}{2}$$

For $\lambda = 2$,

$$(2I - A)\vec{v} = 0 \quad \Rightarrow \quad \begin{bmatrix} \frac{3}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad \Rightarrow \quad -\frac{3}{4}v_1 - \frac{3}{4}v_2 = 0 \quad \Rightarrow \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = \frac{1}{2}$,

$$(\frac{1}{2}I - A)\vec{v} = 0 \quad \Rightarrow \quad \begin{bmatrix} -\frac{3}{4} & -\frac{3}{4} \\ -\frac{3}{4} & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad \Rightarrow \quad -\frac{3}{4}v_1 - \frac{3}{4}v_2 = 0 \quad \Rightarrow \quad \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus, we have found two solutions,

$$\vec{\mathbf{x}}_1(t) = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{\mathbf{x}}_2(t) = e^{\frac{t}{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and the general solution is given by

$$\vec{\mathbf{x}}(t) = C_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{\frac{t}{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

As $t \to \infty$, the second term goes to zero for any C_2. If $C_1 > 0$ then both $x(t), y(t)$ goes to plus infinity and approaches the line $y = x$. If $C_1 < 0$, then the solution goes to negative infinity and approaches the line $y = x$. If $C_1 = 0$, then the solution just approaches $(0, 0)$.