Homework 3

1.3 (1) ($t = 0.1$ and $t = 0.2$ only)

$y' = 3 + t - y$, $y(0) = 1$. Let $f(t,y) = 3 + t - y$, so the equation reads $y' = f(t,y)$

(a) Euler's method with $h = 0.1$:

step n	t_n	y_n	$f(t_n, y_n)$
0	0	1	2
1	0.1	1.2	1.9
2	0.2	1.39	

$\boxed{y(0.1) \approx 1.2}$

$\boxed{y(0.2) \approx 1.39}$

(b) $h = 0.05$:

step n	t_n	y_n	$f(t_n, y_n)$
0	0	1	2
1	0.05	1.1	1.95
2	0.1	1.1975	1.9025
3	0.15	1.2926	1.8574
4	0.2	1.3854	

$\boxed{y(0.1) \approx 1.1975}$

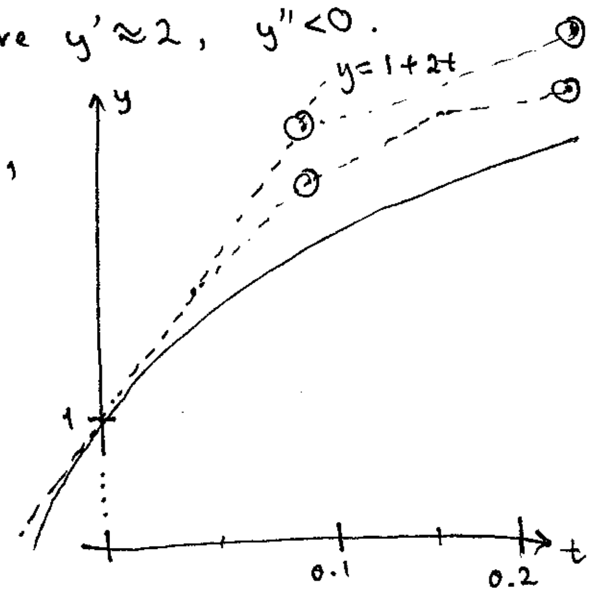
$\boxed{y(0.2) \approx 1.3854}$

Compared to part (a), these values better approximate the solution. To see why, note that

$$y'' = 1 - y'$$

so near the initial value where $y' \approx 2$, $y'' < 0$.

Thus, Euler's method gives overestimates to the solution, which is concave down near $t = 0$.



Bonus problem

Consider the ODE $y' = -200\,y^2$, $y(0) = 1$.

Euler's method $h = 0.01$: Let $f(y) = -200\,y^2$

step n	t_n	y_n	$f(y_n)$
0	0	1	-200
1	0.01	-1	-200
2	0.02	-3	

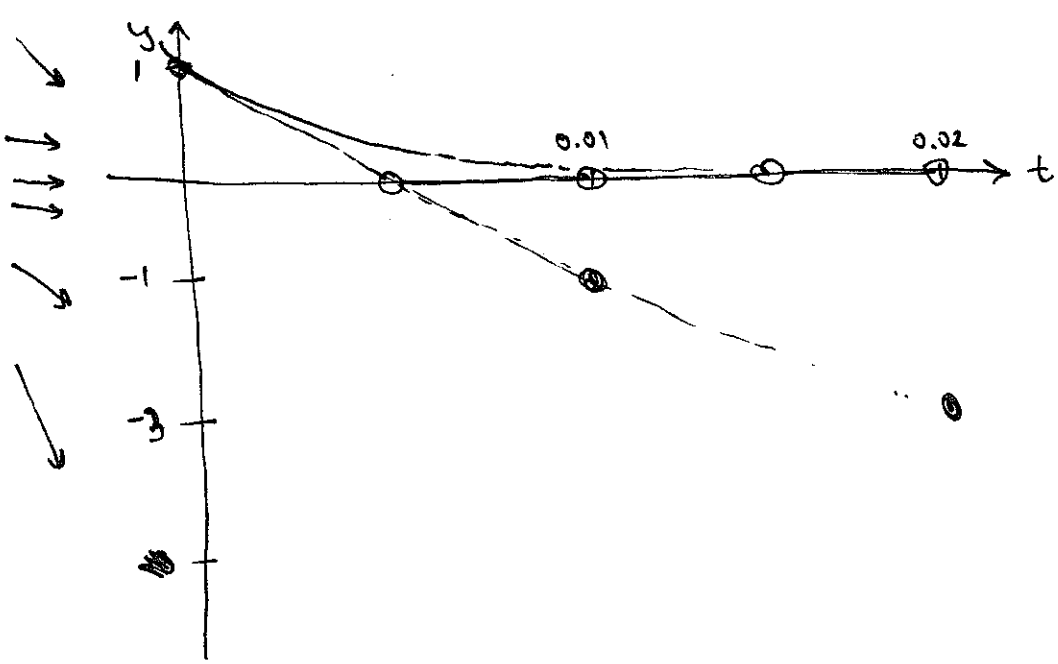
$\boxed{y(0.02) \approx -3}$

$h = 0.005$:

step n	t_n	y_n	$f(y_n)$
0	0	1	-200
1	0.005	0	0
⋮	⋮	⋮	⋮
4	0.02	0	0

$\boxed{y(0.02) \approx 0}$

To explain this, note that for this problem, step sizes of $h = 0.01$ and $h = 0.005$ are large enough to make us overshoot and hit, respectively, the equilibrium solution $y = 0$ (which is semi-stable):



2.5: 3 $\quad (3x^2 - 2xy + 2)\,dx + (6y^2 - x^2 + 3)\,dy = 0$

(a) Exact? $\quad \frac{\partial}{\partial y}(3x^2 - 2xy + 2) = -2x = -2x = \frac{\partial}{\partial x}(6y^2 - x^2 + 3)$

These agree, so the equation is <u>exact</u>.

(b) Solve:

$$\varphi(x,y) = \int (3x^2 - 2xy + 2)\,dx = x^3 - x^2y + 2x + c_1(y)$$

$$\varphi(x,y) = \int (6y^2 - x^2 + 3)\,dy = 2y^3 - x^2y + 3y + c_2(x)$$

These unify to $\quad \varphi(x,y) = x^3 - x^2y + 2x + 2y^3 + 3y$

Thus, the solutions are implicitly given by

$$\underline{x^3 - x^2y + 2x + 2y^3 + 3y = c}$$

7 $\quad (e^x \sin y - 2y \sin x)\,dx + (e^x \cos y + 2\cos x)\,dy = 0$

(a) Exact? $\quad \frac{\partial}{\partial y}(e^x \sin y - 2y \sin x) = e^x \cos y - 2\sin x = e^x \cos y - 2\sin x = \frac{\partial}{\partial x}(e^x \cos y + 2\cos x)$

These agree, so the equation is <u>exact</u>.

(b) Solve:

$$\varphi(x,y) = \int (e^x \sin y - 2y \sin x)\,dx = e^x \sin y + 2y\cos x + c_1(y)$$

$$\varphi(x,y) = \int (e^x \cos y + 2\cos x)\,dy = e^x \sin y + 2y\cos x + c_2(x)$$

These unify to $\quad \varphi(x,y) = e^x \sin y + 2y\cos x$

Thus, the solutions are implicitly given by

$$\underline{e^x \sin y + 2y \cos x = c}$$

Note that for $c=0$ we find in particular the equilibrium solution <u>$y=0$</u>.

2.5: 17 Assume M, N, M_y, N_x are continuous in the rectangular region R, and assume $M_y = N_x$ on R. We must show that

$$\psi(x,y) = \int_{x_0}^{x} M(s,y_0)\,ds + \int_{y_0}^{y} N(x,t)\,dt ,$$

where (x_0,y_0) is a chosen point in R, satisfies $\psi_x = M$ and $\psi_y = N$ on R.

We calculate:

$$\psi_x(x,y) = \frac{\partial}{\partial x}\int_{x_0}^{x} M(s,y_0)\,ds + \frac{\partial}{\partial x}\int_{y_0}^{y} N(x,t)\,dt$$

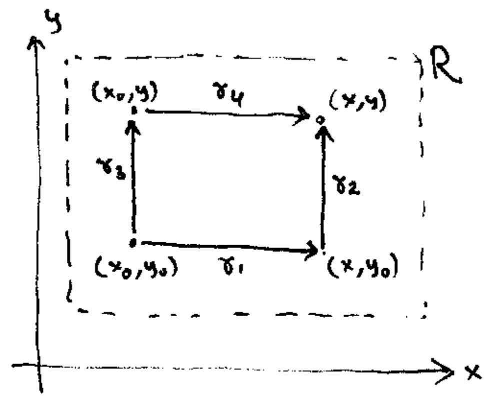
$$= M(x,y_0) + \int_{y_0}^{y} N_x(x,t)\,dt \qquad \left(\text{FTC1; swap } \tfrac{\partial}{\partial x} \textstyle\int\right)$$

$$= M(x,y_0) + \int_{y_0}^{y} M_y(x,t)\,dt \qquad (N_x = M_y)$$

$$= M(x,y_0) + \Big[M(x,t)\Big]_{t=y_0}^{y} = \underline{M(x,y)} \ \checkmark \qquad (\text{FTC2})$$

$$\psi_y(x,y) = 0 + \frac{\partial}{\partial y}\int_{y_0}^{y} N(x,t)\,dt = \underline{N(x,y)} \ \checkmark \qquad (\text{FTC1})$$

Alternatively, if you've taken Math 52:



The 1-form $\omega = M\,dx + N\,dy$ is exact, so

$$\psi(x,y) = \int_{x_0}^{x} M(s,y_0)\,ds + \int_{y_0}^{y} N(x,t)\,dt$$

$$= \int_{\gamma_1}\omega + \int_{\gamma_2}\omega$$

$$\overset{\text{Green's Thm}}{=} \int_{\gamma_3}\omega + \int_{\gamma_4}\omega$$

$$= \int_{y_0}^{y} N(x_0,t)\,dt + \int_{x_0}^{x} M(s,y)\,ds$$

From the last line it is clear that $\psi_x = M$, while from the first line it is clear that $\psi_y = N$.

2.5: 19 $x^2y^3 + x(1+y^2)\,y' = 0$

(a) Not exact: $\frac{\partial}{\partial y}(x^2y^3) = 3x^2y^2 \neq 1+y^2 = \frac{\partial}{\partial x}(x(1+y^2))$

Apply integrating factor $\mu(x,y) = \frac{1}{xy^3}$:

$$x\,dx + (y^{-3} + y^{-1})\,dy = 0$$

This equation is obviously exact, indeed separable.

(b) To solve, separate:

$$(y^{-3} + y^{-1})\,dy = -x\,dx,$$

$$-\tfrac{1}{2}y^{-2} + \ln|y| = -\tfrac{1}{2}x^2 + c,$$

$$\underline{x^2 - y^{-2} + 2\ln|y| = c} \qquad (\text{new } c!)$$

But: μ is undefined for $y=0$, which satisfies the original equation, so we get the equilibrium solution $\underline{y=0}$ as well.

23 If $\frac{N_x - M_y}{M} = Q(y)$ a function of y alone, then we must check that $\mu = \exp\int Q(y)\,dy$ is an integrating factor for the equation $M + Ny' = 0$:

$$\frac{\partial}{\partial y}(M\cdot\mu) = M_y\cdot\mu + MQ\mu = (M_y + N_x - M_y)\mu$$

$$= N_x\mu = \frac{\partial}{\partial x}(N\mu). \quad \text{It works!}$$

2.5: 25 (a) $(3x^2y + 2xy + y^3)dx + (x^2+y^2)dy = 0$

This equation is not exact, as

$$\frac{\partial}{\partial y}(3x^2y + 2xy + y^3) = 3x^2 + 2x + 3y^2$$
$$\neq 2x = \frac{\partial}{\partial x}(x^2+y^2).$$

An integration factor $\mu = \mu(x,y)$ needs to unify

$$\frac{\partial}{\partial y}\left((3x^2y+2xy+y^3)\cdot\mu\right)$$
$$= (3x^2+2x+3y^3)\cdot\mu + (3x^2y+2xy+y^3)\cdot\mu_y$$

and $\frac{\partial}{\partial x}\left((x^2+y^2)\cdot\mu\right)$

$$= (2x)\cdot\mu + (x^2+y^2)\cdot\mu_x$$

Moving everything to one side, we need

$$3(x^2+y^2)\cdot\mu - (x^2+y^2)\cdot\mu_x + (3x^2y+2xy+y^3)\mu_y = 0$$

We see that having $\mu_y = 0$ and $\mu_x = 3\mu$ works, i.e., $\underline{\mu = e^{3x}}$.

Now we integrate:

$$\varphi(x,y) = \int (3x^2y + 2xy + y^3)e^{3x}\,dx$$
$$= \left(x^2 - \tfrac{2}{3}x + \tfrac{2}{9}\right)e^{3x}y + \left(\tfrac{2}{3}x - \tfrac{2}{9}\right)e^{3x}y + \tfrac{1}{3}y^3e^{3x} + c_1(y)$$
$$\varphi(x,y) = \int (x^2+y^2)e^{3x}\,dy = \tfrac{1}{3}y^3e^{3x} + c_2(x) + x^2ye^{3x}$$

$$\int xe^{3x}dx = \tfrac{1}{3}xe^{3x} - \tfrac{1}{3}\int e^{3x}dx = \tfrac{1}{3}xe^{3x} - \tfrac{1}{9}e^{3x}.$$

$$\int 3x^2e^{3x}dx = x^2e^{3x} - \int 2xe^{3x}dx = x^2e^{3x} - \tfrac{2}{3}xe^{3x} + \tfrac{2}{9}e^{3x}.$$

These are unified by

$$\varphi(x,y) = \tfrac{1}{3}y^3e^{3x} + yx^2e^{3x}.$$

Solutions: $\underline{(y^3 + 3yx^2)e^{3x} = \mathbb{C}}$

2.3: (2) $t(t-4)y' + y = 0$, $y(2)=1$

Divide by $t(t-4)$ to get

$$y' + \frac{1}{t(t-4)} y = 0.$$

The coefficients in this linear ODE are continuous on $(-\infty,0) \cup (0,4) \cup (4,+\infty)$

By Theorem 2.3.1, the solution exists on the part containing the initial condition, $\underline{(0,4)}$.

(4) $y'(4-t^2) + 2t \cdot y = 3t^2$, $y(-3)=1$

In normal form: $y' + \frac{2t}{4-t^2} y = \frac{3t^2}{4-t^2}$

Coefficients are continuous on $(-\infty,-2) \cup (-2,2) \cup (2,+\infty)$

The initial condition $y(-3)=1$ yields via Thm 2.3.1 the interval of existence: $\underline{(-\infty,-2)}$

(13) $y' = -\frac{4t}{y}$, $y(0)=y_0$. Separable:

$y\,dy = -4t\,dt$, $\frac{1}{2}y^2 = -2t^2 + c$.

$y(0)=y_0 \Rightarrow c = \frac{1}{2}y_0^2$. So $\underline{y = \pm\sqrt{y_0^2 - 4t^2}}$ (taking +/- to ensure $y(0)=y_0$)

Must have $y \neq 0$, so we require $y_0 \neq 0$. Then the interval of existence is where $|y_0^2 - 4t^2 > 0$

$\Leftrightarrow$ $\underline{|t| < \frac{1}{2}|y_0|}$.

2.4 : 14 $\quad y' = 2ty^2, \qquad y(0) = y_0.$

Note that $\underline{y=0}$ is an equilibrium solution,

So $y_0 = 0$ yields IoE (Interval of Existence): $\underline{(-\infty, \infty)}$

For $y_0 \neq 0$ we separate:

$$y^{-2}\,dy = 2t\,dt, \quad -y^{-1} = t^2 + c$$

$$y(0) = y_0 \Rightarrow c = -y_0^{-1}$$

So the solution is $\underline{y = (y_0^{-1} - t^2)^{-1}}$

For $y_0 < 0$, this is defined for all t, IoE $= \underline{(-\infty, \infty)}$

For $y_0 > 0$, we must have $t^2 \neq y_0^{-1}$, $t \neq 1/\sqrt{y_0}$,

and we must take the interval containing 0,

giving $\underline{\left(-1/\sqrt{y_0},\ 1/\sqrt{y_0}\right)}$