Problem Set 2.

2.1.1

$$y' = \frac{x^2}{y} \iff y\,dy = x^2\,dx.$$

$$\Rightarrow \int y\,dy = \int x^2\,dx \Rightarrow \frac{y^2}{2} = \frac{x^3}{3} + C.$$

2.1.3

$$y' + y^2 \sin x = 0 \iff \frac{dy}{y^2} = -\sin x\,dx.$$

$$\int \frac{dy}{y^2} = \int -\sin x\,dx \qquad -\frac{1}{y} = \cos x + C.$$

$$\iff \cos x + \frac{1}{y} = 0. \qquad (y \neq 0).$$

Also, $y = 0$ is also a solution.

2.1.5.

$$y' = (\cos^2 x)(\cos^2(2y)) \iff \frac{dy}{\cos^2(2y)} = \cos^2 x\,dx$$

$$\iff \int \frac{dy}{\cos^2(2y)} = \int \cos^2 x\,dx.$$

$$\iff \frac{1}{2}\tan(2y) = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$$

$$\iff \tan(2y) = x + \frac{\sin 2x}{2} + C. \quad \left(y \neq \frac{2k+1}{4}\pi,\ k \in \mathbb{Z}\right).$$

Notice, $y = \frac{2k+1}{4}\pi$, $k \in \mathbb{Z}$ are also solutions.

2.1.8

$$\frac{dy}{dx} = \frac{x^2}{1+y^2} \iff (1+y^2)\,dy = x^2\,dx$$
$$\iff \int (1+y^2)\,dy = \int x^2\,dx.$$
$$\iff y + \frac{y^3}{3} = \frac{x^3}{3} + C.$$

2.1.15.

$$\begin{cases} y' = \dfrac{2x}{1+2y} \\ y(2) = 0 \end{cases}$$

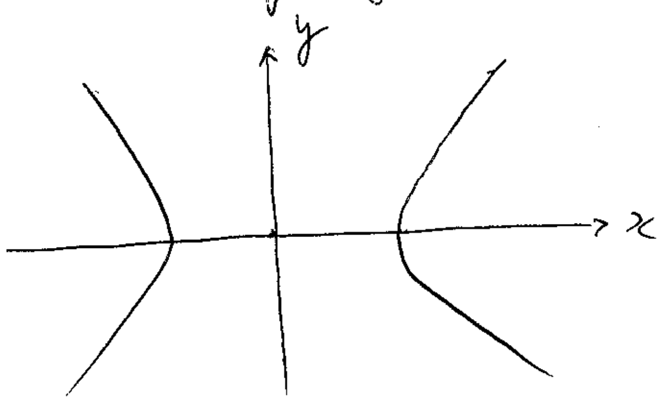
$$(1+2y)\,dy = 2x\,dx$$
$$\int (1+2y)\,dy = \int 2x\,dx$$
$$y + y^2 = x^2 + C.$$

Plugin $(y, x) = (0, 2)$, we have

$$0 = 4 + C \Rightarrow C = -4.$$

Therefore, the solution to the initial value problem is

$$y + y^2 = x^2 - 4 \qquad x^2 - \left(y+\tfrac{1}{2}\right)^2 = \frac{15}{4}$$



$$\left(-\infty, -\frac{\sqrt{15}}{2}\right] \cup \left[\frac{\sqrt{15}}{2}, +\infty\right)$$

2.5.30

$$\frac{dy}{dx} = \frac{y-4x}{x-y}$$

(a) $$\frac{dy}{dx} = \frac{(y-4x)\frac{1}{x}}{(x-y)\frac{1}{x}} = \frac{(y/x)-4}{1-(y/x)}$$

(b) $$\begin{cases} v = \frac{y}{x} \\ y = xv \end{cases} \Longrightarrow \frac{dy}{dx} = \frac{d(xv)}{dx} = x\frac{dv}{dx} + v$$

(c) By (a) and (b), $$v + x\frac{dv}{dx} = \frac{v-4}{1-v}$$

$$\Longleftrightarrow x\frac{dv}{dx} = \frac{v^2-4}{1-v}$$

(d) $$\frac{1-v}{v^2-4}\,dv = \frac{dx}{x} \Longleftrightarrow \int \frac{1-v}{v^2-4}\,dv = \int \frac{dx}{x}$$

$$\Longleftrightarrow \int \left(-\frac{3}{4}\right)\frac{dv}{v+2} + \int\left(-\frac{1}{4}\right)\frac{dv}{v-2} = \int \frac{dx}{x}$$

$$\Longleftrightarrow \ln|x| = -\frac{3}{4}\ln|v+2| - \frac{1}{4}\ln|v-2| + C$$

$$\Longleftrightarrow x = (v+2)^{-\frac{3}{4}}(v-2)^{-\frac{1}{4}}\cdot C_0$$

(e) By (d) and $v = \frac{y}{x}$, we have

$$x = \left(\frac{y}{x}+2\right)^{-\frac{3}{4}}\left(\frac{y}{x}-2\right)^{-\frac{1}{4}} C_0$$

$$\Longleftrightarrow (y-2x)(y+2x)^3 = C_1$$

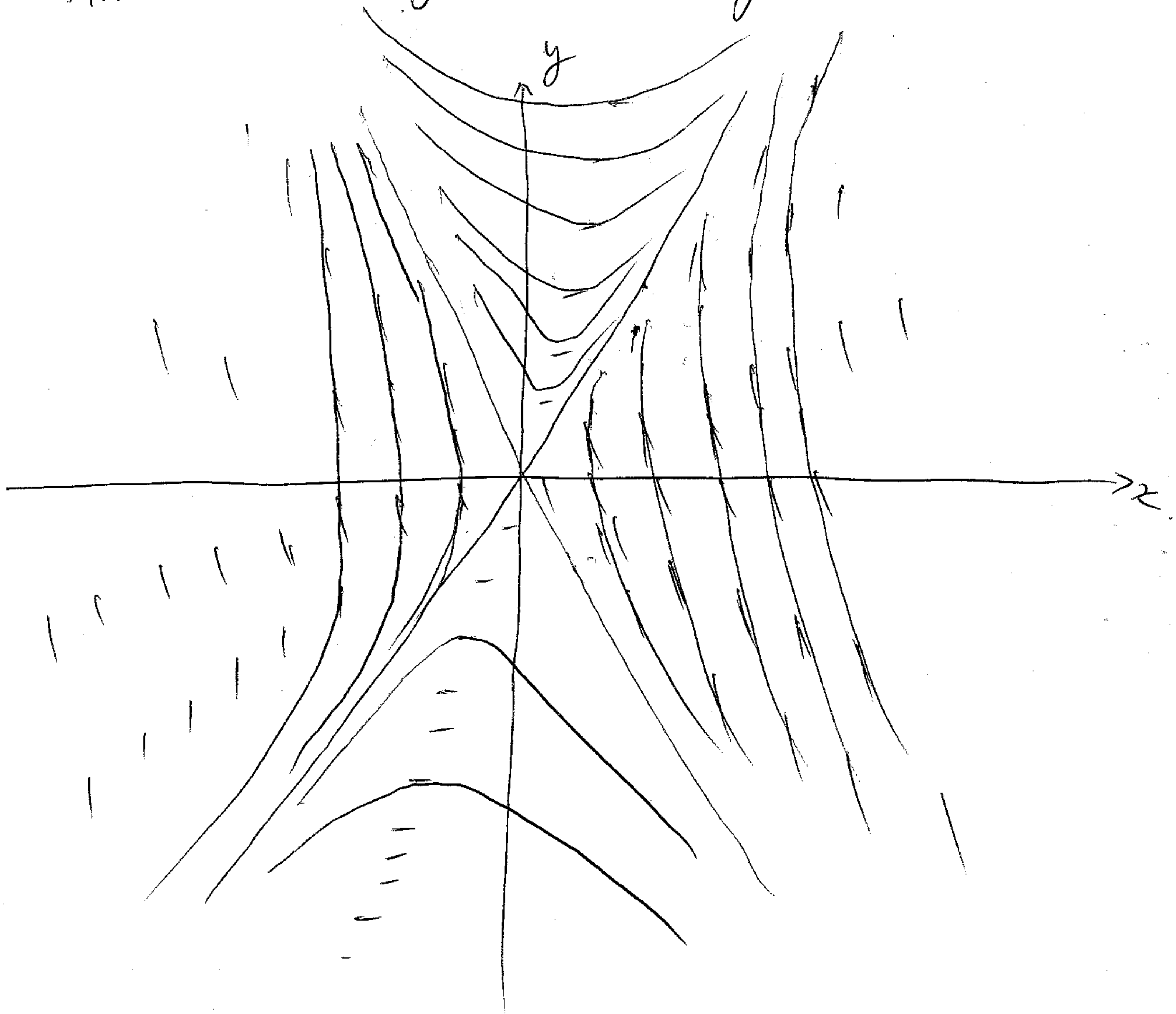
f). On the line $y=4x$, the slopes are zeros.

On the line $y=x$, the slopes are infinities.

When $4 > \frac{y}{x} > 1$, the slopes are positive.

In other places, the slopes are negative.

Also notice that, by choosing $C_1 = 0$, we have two solutions $y=2x$ and $y=-2x$



2.2.17

$$\begin{cases} \dfrac{du}{dt} = -\alpha u^4 \\ u(0) = 2000 \end{cases}$$

(a) $\dfrac{du}{u^4} = -\alpha dt \iff \displaystyle\int \frac{du}{u^4} = \int (-\alpha) dt$

$$\iff (-\tfrac{1}{3}) u^{-3} = -\alpha t + C.$$

$$\iff u = \frac{1}{\sqrt[3]{3\alpha t + C_1}}$$

By $u(0) = 2000$, we have

$$2000 = \frac{1}{\sqrt[3]{C_1}}$$

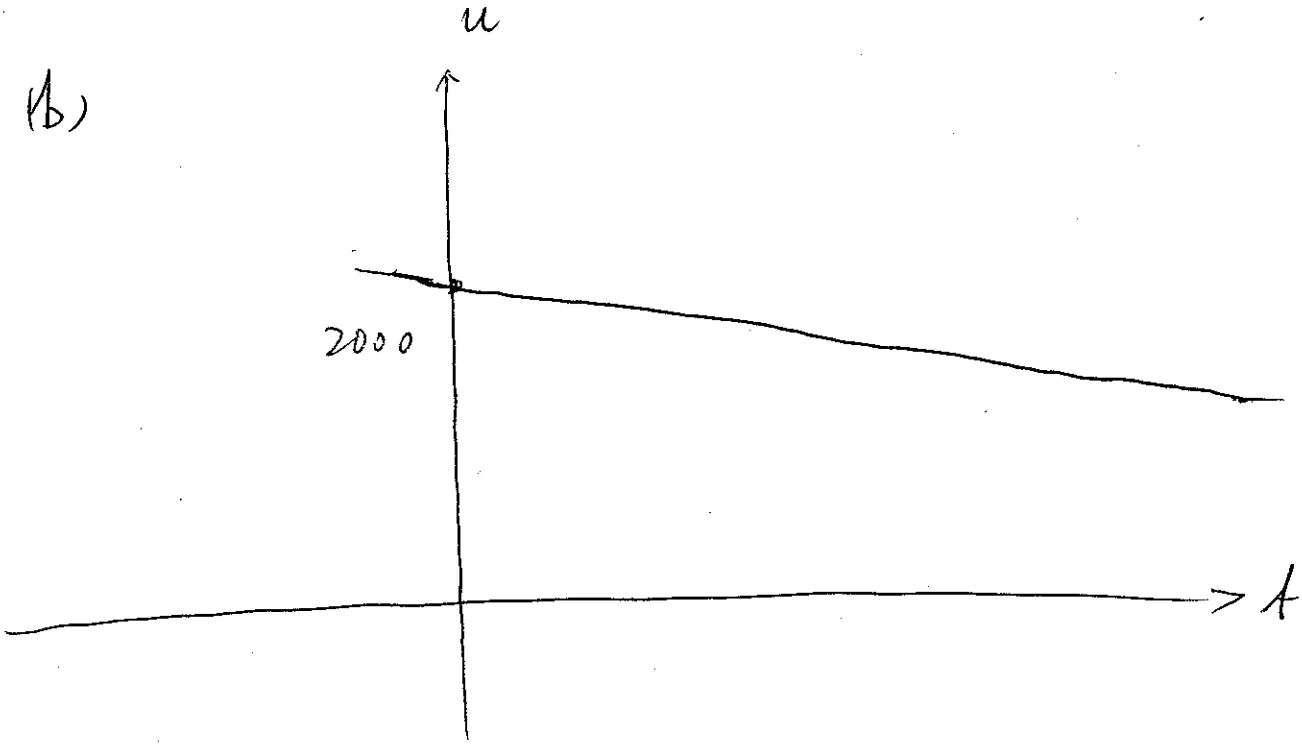
$$\iff C_1 = 1/(8 \times 10^9) = 125 \times 10^{-12}.$$

Then the solution is

$$u(t) = \frac{1}{\sqrt[3]{6 \times 10^{-12} t + 125 \times 10^{-12}}}$$

$$= \frac{10^4}{\sqrt[3]{6t + 125}} = 2000/(1 + 0.048t)^{\frac{1}{3}}$$

(b)



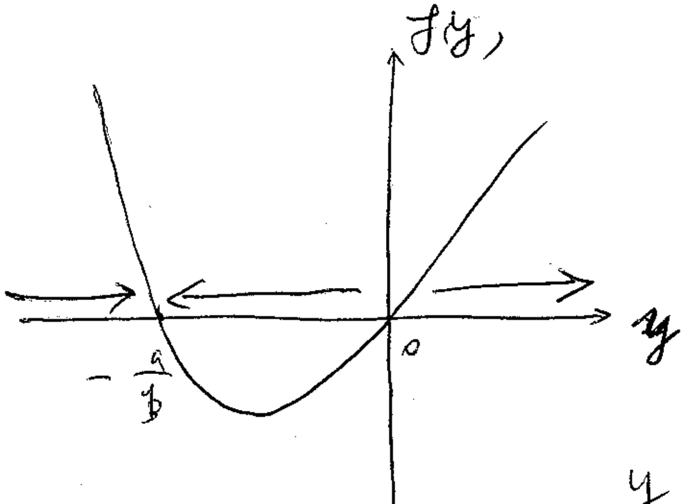
(c) $u(\tau) = \dfrac{2000}{(1+0.048\tau)^{\frac{1}{3}}} = 600$

$\Longleftrightarrow \quad \tau = 750.77\ (s).$

2.4.1

$$\begin{cases} \dfrac{dy}{dt} = ay + by^2 \\ y_0 \geq 0 \end{cases} \qquad a > 0,\ b > 0$$

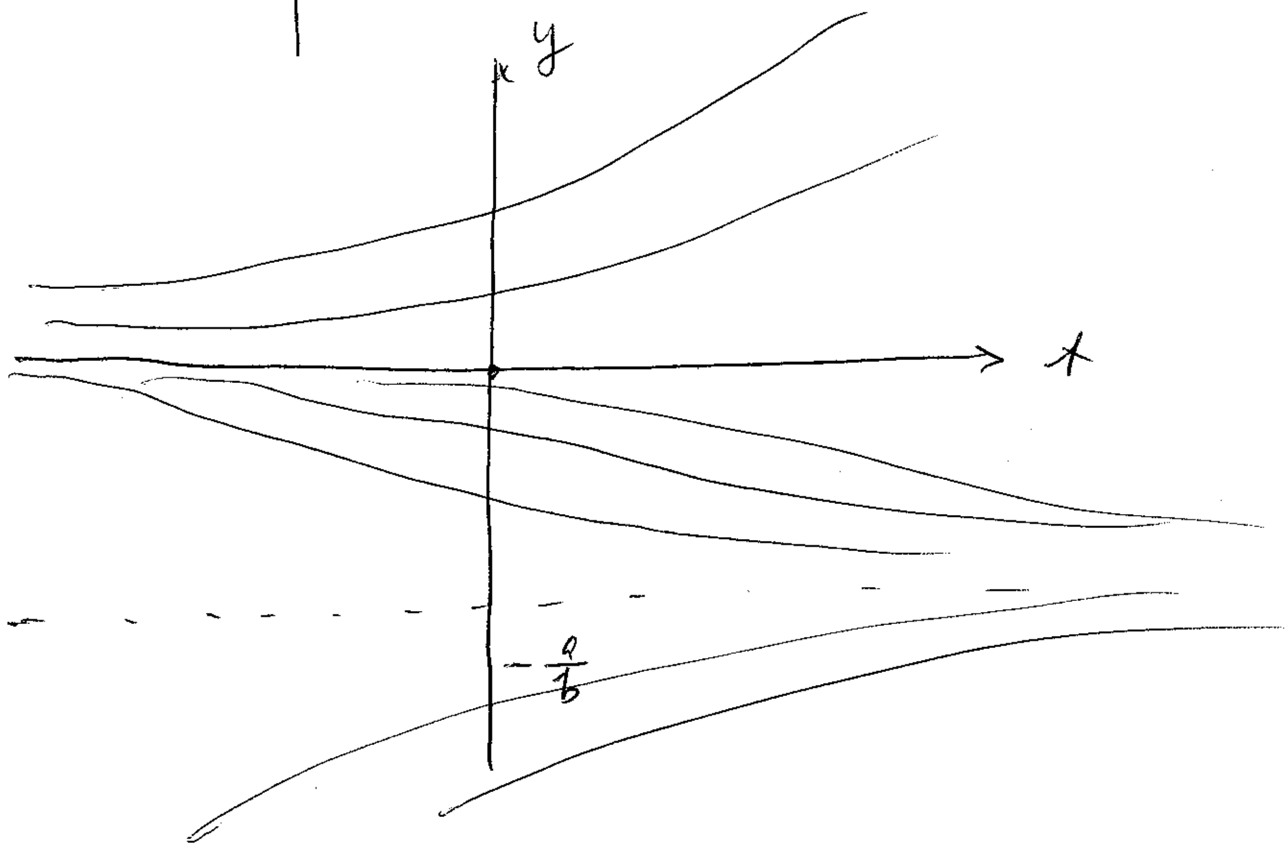
$$f(y) = ay + by^2 = ay\left(1 + \frac{b}{a}y\right) = by\left(y + \frac{a}{b}\right)$$



critical points

0: unstable

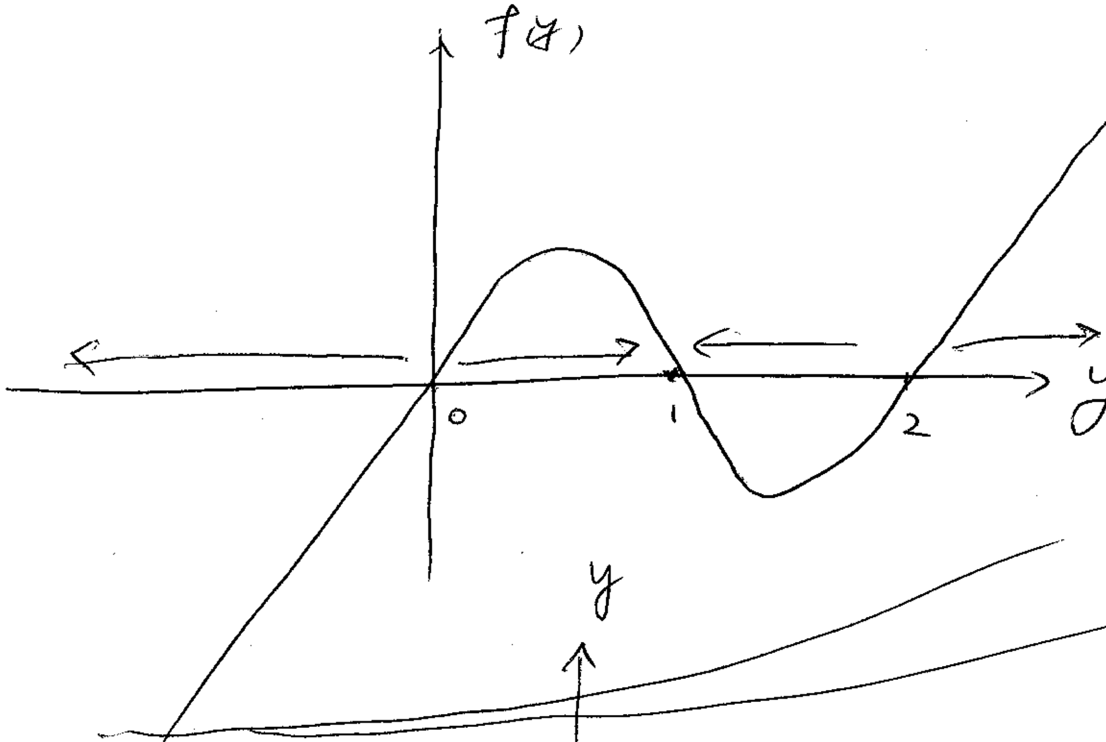
$-\frac{a}{b}$: stable



2.4.3

$$\begin{cases} \dfrac{dy}{dt} = y(y-1)(y-2) \\ y_0 \geq 0 \end{cases}$$

$$f(y) = y(y-1)(y-2)$$

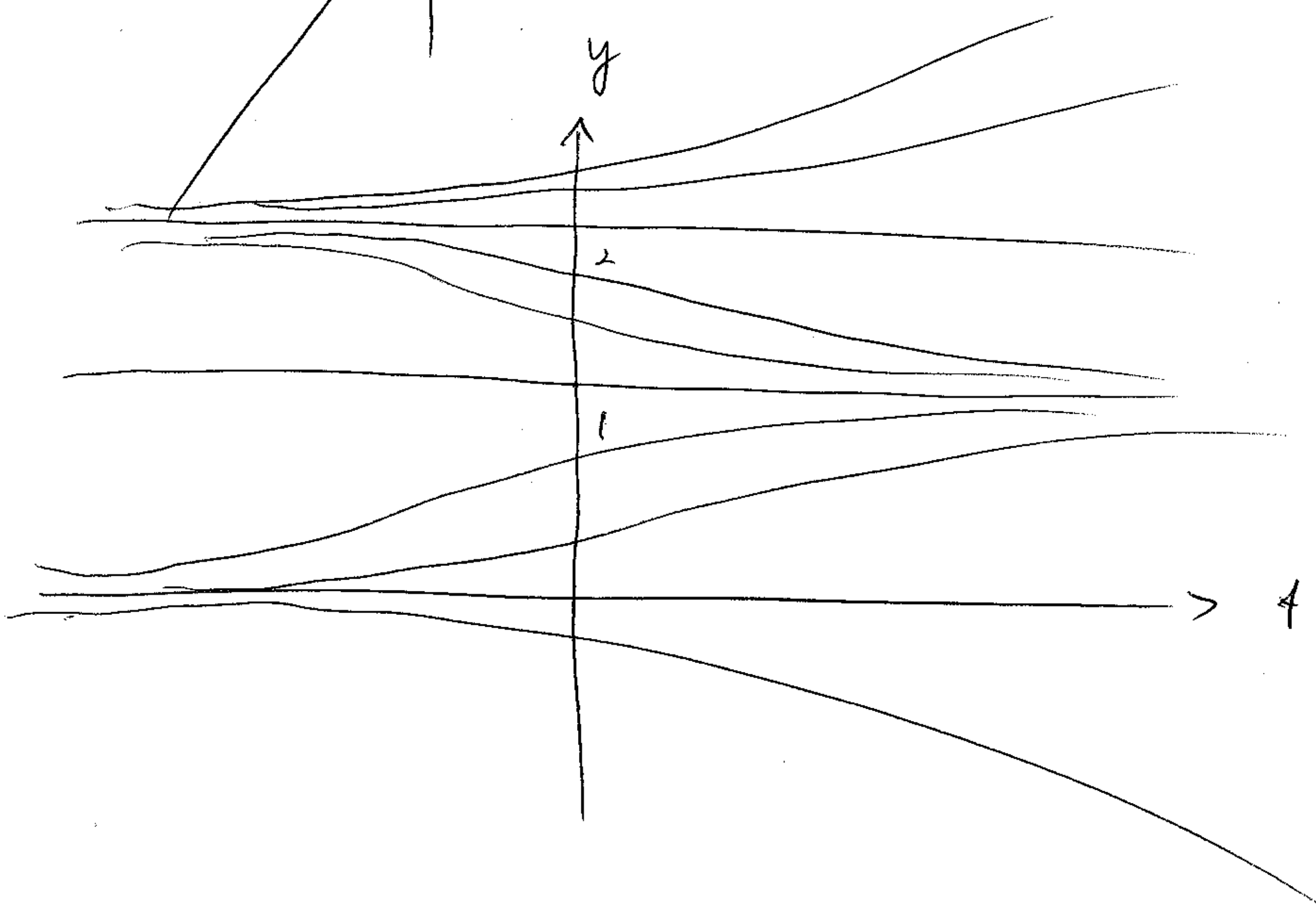


critical points:

0 : unstable

1 : stable

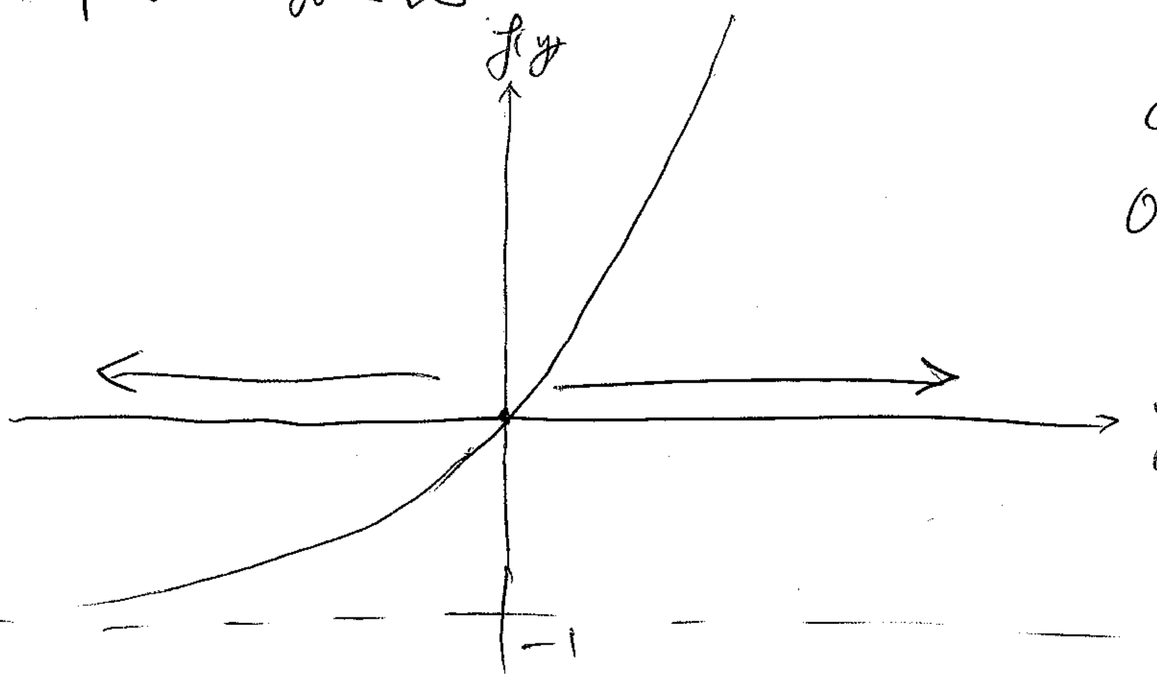
2 : unstable



2.4.4

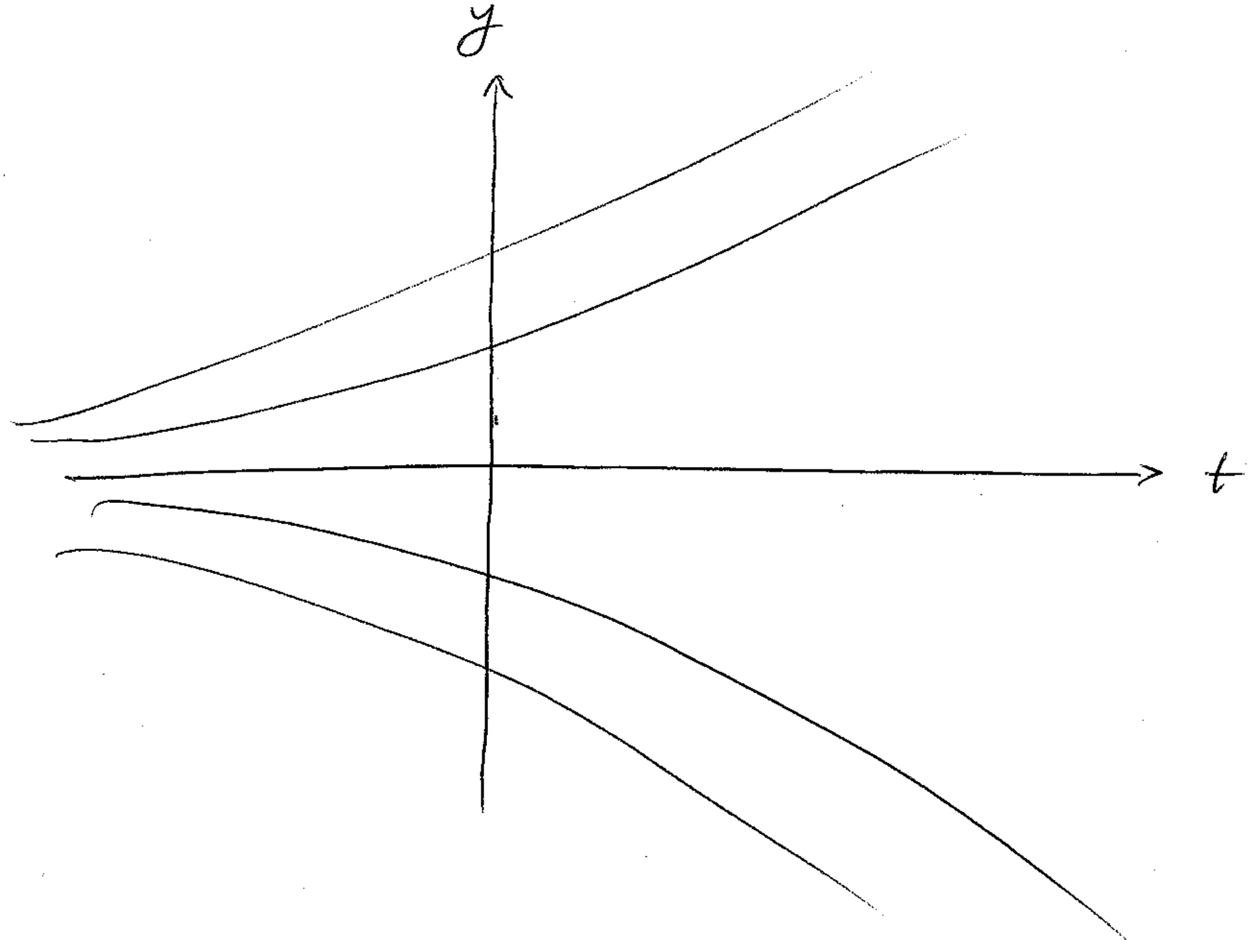
$$\begin{cases} \dfrac{dy}{dt} = e^y - 1 \\ -\infty < y_0 < \infty \end{cases}$$

$f(y) = e^y - 1$



critical point:

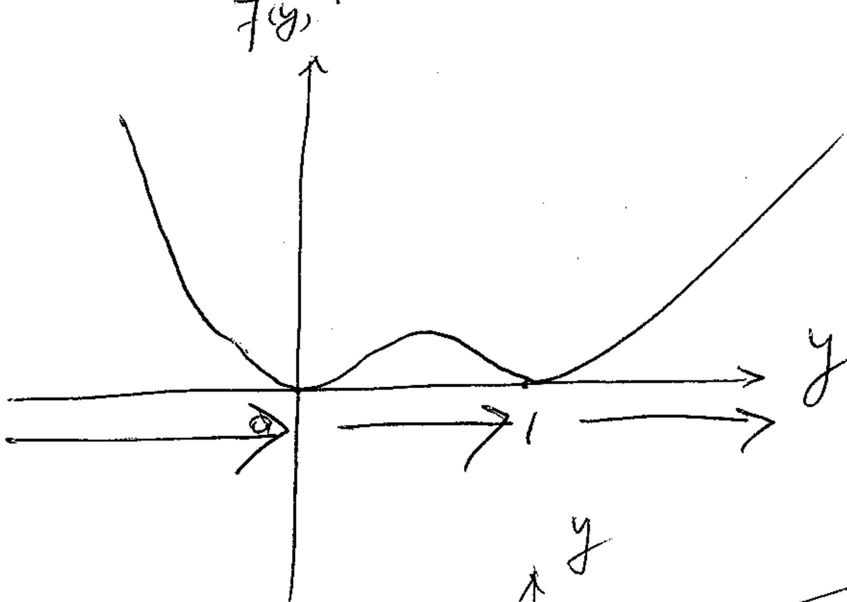
0: unstable



2.4.13

$$\begin{cases} \dfrac{dy}{dt} = y^2(1-y)^2 \\ -\infty < y_0 < +\infty \end{cases}$$

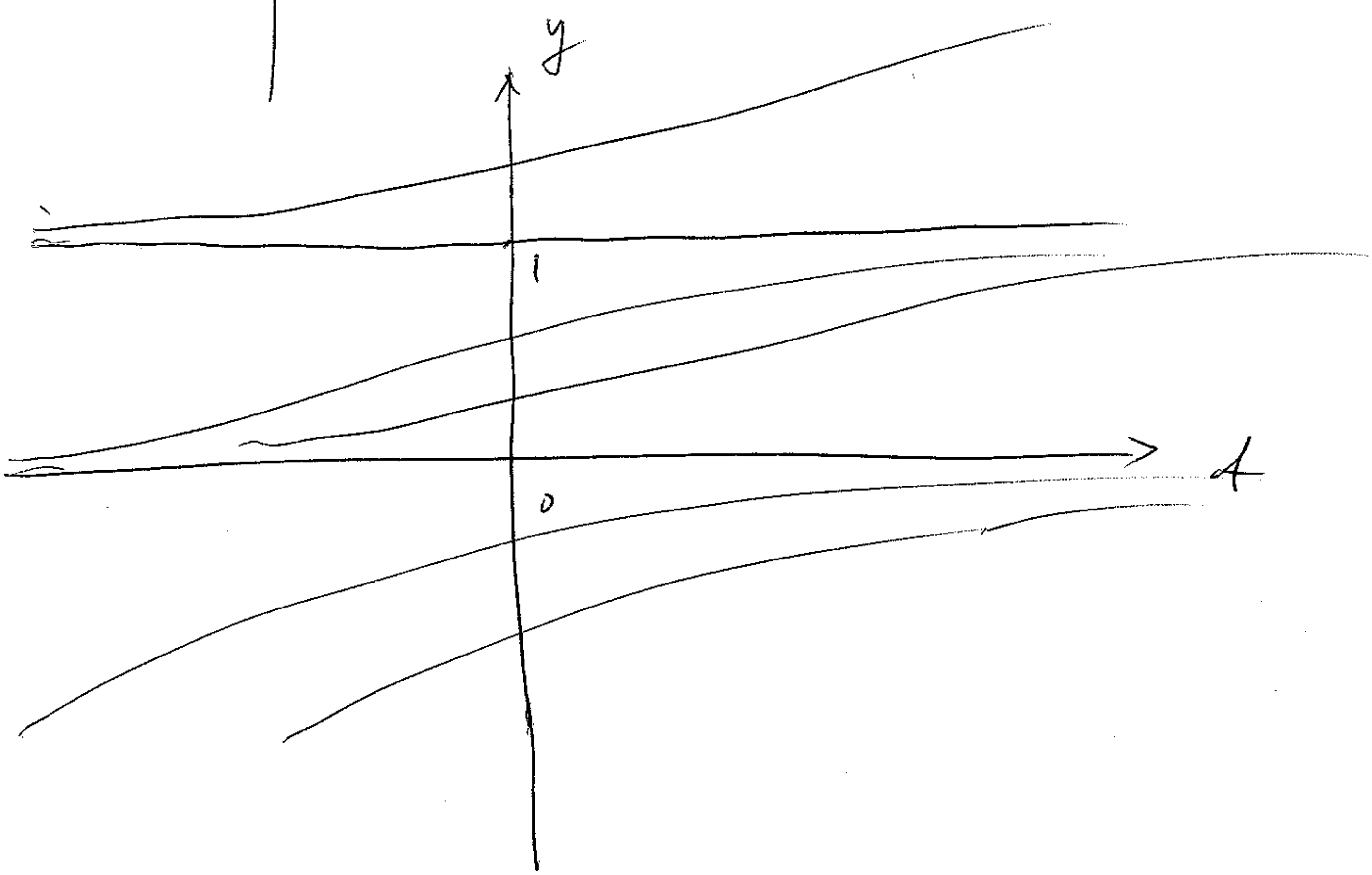
$$f(y) = y^2(1-y)^2.$$



critical points.

0: semistable

1: semistable.



2. 4. 14

$$\frac{dy}{dt} = ry\left[1 - \frac{y}{K}\right].$$

(a) $\begin{cases} y_0 = \dfrac{K}{3} \\ y(\tau) = \dfrac{2K}{3}. \end{cases}$

$$\frac{dy}{y\left(1-\frac{y}{K}\right)} = r\,dt$$

$$\Longleftrightarrow \int \left(\frac{1}{y} + \frac{1}{K-y}\right) dy = \int r\,dt$$

$$\Longleftrightarrow \ln|y| - \ln|K-y| = rt + C$$

$$\Longleftrightarrow \frac{y}{K-y} = C_1 e^{rt}$$

$$\Longleftrightarrow y = \frac{C_1 K e^{rt}}{1 + C_1 e^{rt}}.$$

$$y(0) = \frac{K}{3} = \frac{C_1 K}{1 + C_1} \Longrightarrow C_1 = \frac{1}{2}.$$

Then $y(t) = \dfrac{Ke^{rt}}{2 + e^{rt}}$

$$y(\tau) = \frac{2K}{3} \Longrightarrow \frac{Ke^{r\tau}}{2 + e^{r\tau}} = \frac{2K}{3}$$

$$\Longrightarrow \tau = \frac{1}{r}\ln 4 \approx 55.45 \text{ (year)}$$

(b) We still have

$$y = \frac{C_1 K e^{rt}}{1 + C_1 e^{rt}}$$

$$\begin{cases} y(0) = \alpha K & \Rightarrow \quad \dfrac{C_1}{1+C_1} = \alpha \Rightarrow C_1 = \dfrac{\alpha}{1-\alpha} \\ y(T) = \beta K . \end{cases}$$

$$y(T) = \frac{\frac{\alpha}{1-\alpha} K e^{rT}}{1 + \frac{\alpha}{1-\alpha} e^{rT}} = \beta K$$

$$\Rightarrow \quad T = \frac{1}{r} \ln\left(\frac{\beta(1-\alpha)}{(1-\beta)\alpha}\right).$$

When $\begin{cases} \alpha = 0.1, \\ \beta = 0.9 \end{cases}$ $\quad T = \cancel{178}.\ 175.78$ years.