

MATH 53 HOMEWORK 1 SOLUTION

Section 1.1 problems 6, 9, 15, 32, 34, 39

Section 1.2 problems 5, 15, 17, 20, 28

Problem 1.1.6 The directional field for $y' = y + 2$ is given by Figure 1.1.11 in the text. From the plot, one sees that if $y(0) > -2$, then $y(t) \rightarrow \infty$ as $t \rightarrow \infty$, if $y(0) = -2$, then $y(t) = -2$ for all t , and if $y(0) < -2$, then $y(t) \rightarrow -\infty$ as $t \rightarrow \infty$.

Problem 1.1.9 We are looking for $y' = ay + b$ with $2a + b = 0$, $ay + b > 0$ when $y > 2$ and $ay + b < 0$ when $y < 2$. So a must be positive and $b = -2a$. For example,

$$\frac{dy}{dt} = y - 2$$

Problem 1.1.15 Denote the corresponding ODE by $y' = f(y)$. From figure 1.1.8, we know that 2 is an equilibrium point, so $f(2) = 0$. Also, the slope is decreasing as y increases, so $f(y)$ is a decreasing function. The only ODE from the given choices that matches these two properties is (j) $y' = 2 - y$.

Problem 1.1.32 (a) We multiply by the integrating factor $e^{-0.5t}$ to get

$$e^{-0.5t}p'(t) - 0.5e^{-0.5t}p(t) = -450e^{-0.5t}$$

so

$$\frac{d}{dt}(e^{-0.5t}p(t)) = -450e^{-0.5t}$$

Thus,

$$e^{-0.5t}p(t) = \int -450e^{-0.5t} = 900e^{-0.5t} + C \Rightarrow p(t) = 900 + Ce^{0.5t}$$

Plugging $t = 0, p(0) = 850$ gives

$$850 = 900 + C \cdot 1 \Rightarrow C = -50$$

so

$$p(t) = 900 - 50e^{0.5t}$$

Denote the time of extinction by T , then $p(T) = 0$, so

$$0 = 900 - 50e^{0.5T} \Rightarrow e^{0.5T} = \frac{900}{50} = 18 \Rightarrow T = 2 \ln 18 \approx 5.78 \text{ months}$$

(b) We know from part (a) that the general solution is

$$p(t) = 900 + Ce^{0.5t}$$

Plugging $t = 0, p(0) = p_0$ gives

$$p_0 = 900 + C \Rightarrow C = p_0 - 900$$

so

$$p(t) = 900 + (p_0 - 900)e^{0.5t}$$

Plugging $t = T, p(T) = 0$ gives

$$0 = 900 + (p_0 - 900)e^{0.5T} \Rightarrow T = 2 \ln \left(\frac{900}{900 - p_0} \right)$$

(c) From (b), we have

$$T = 2 \ln \left(\frac{900}{900 - p_0} \right)$$

Let $T = 12$, so

$$12 = 2 \ln \left(\frac{900}{900 - p_0} \right) \Rightarrow \frac{900}{900 - p_0} = e^6 \Rightarrow p_0 = 900(1 - e^{-6}) \approx 897.8$$

Problem 1.1.34 (a) From

$$Q'(t) = -rQ \Rightarrow Q(t) = Q(0)e^{-rt} = 100e^{-rt}$$

The assumption tells us that $Q(7) = 82.04$ so

$$82.04 = 100e^{-7r} \Rightarrow r = -\frac{1}{7} \ln \frac{82.04}{100} \approx 0.02828 \text{ day}^{-1}$$

(b) From part (a),

$$Q(t) = 100e^{-rt} = 100e^{-0.02828t}$$

(c) We want to find t such that $Q(t) = Q(0)/2$, so

$$Q(0)e^{-0.02828t} = \frac{Q(0)}{2} \Rightarrow t = -\frac{1}{0.02828} \ln 0.5 \approx 24.51 \text{ days}$$

Problem 1.1.39 (a) Let $Q(t)$ denote the amount of drug (in mg) in the bloodstream at time t , then

$$Q(t + \Delta t) - Q(t) = Q_{in} - Q_{out} = 5 \times 100 \times \Delta t - 100 \times Q(t) \times \Delta t = 100\Delta t(5 - Q(t))$$

Divide by Δt and take $\Delta t \rightarrow 0$, we have

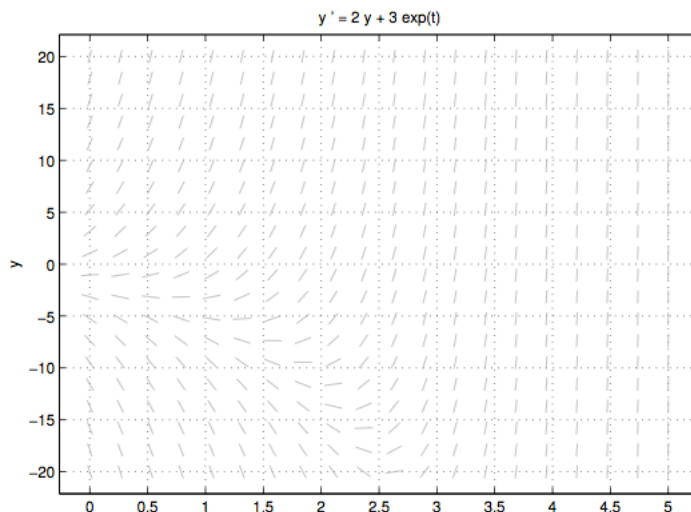
$$Q'(t) = 100(5 - Q(t))$$

(b) From the ODE derived in (a), the general solution (see Eq (2) in the text) is

$$Q(t) = 5 + Ce^{-100t}$$

Thus, $Q(t) \rightarrow 5$ as $t \rightarrow \infty$.

Problem 1.2.5 (a) The plot is



(b) From the plot, one can guess that the solution grow exponentially fast for large t . From (c), we will see that this is not always the case and, in fact, there is a region where it actually decreases exponentially. However, this might not be easy to observe from the plot.

(c) Multiply by integrating factor e^{-2t} to get

$$(e^{-2t}y)' = 3e^{t-2t} = 3e^{-t} \Rightarrow e^{-2t}y = -3e^{-t} + C \Rightarrow y(t) = -3e^t + Ce^{2t}$$

So $y \rightarrow \infty$ or $-\infty$ as $t \rightarrow \infty$ depending on the sign of C .

Problem 1.2.15 Divide by t to get

$$y'(t) + \frac{2}{t}y(t) = t - 1 + \frac{1}{t}$$

Multiply by the integrating factor $e^{\int 2/t dt} = e^{2 \ln |t|} = t^2$,

$$t^2 y' + 2ty = t^3 - t^2 + t \Rightarrow t^2 y = \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + C \Rightarrow y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{C}{t^2} = \frac{3t^4 - 4t^3 + 6t^2 + C'}{12t^2}$$

Plugging in initial condition $y(1) = 1/2$ and solve for C' to get $C' = 1$. So the solution is

$$y(t) = \frac{3t^4 - 4t^3 + 6t^2 + 1}{12t^2}$$

Problem 1.2.17 Multiply by integrating factor e^{-2t} to get

$$(e^{-2t}y)' = 1 \Rightarrow e^{-2t}y = t + C \Rightarrow y = e^{2t}(t + C)$$

Plugging in initial condition $y(0) = 2$ to get $C = 2$, so the solution is given by

$$y'(t) = (t + 2)e^{2t}$$

Problem 1.2.20 Divide by t to get

$$y' + (1 + \frac{1}{t})y = 1$$

The integrating factor is

$$e^{\int 1 + \frac{1}{t} dt} = e^{t + \ln |t|} = te^t$$

Multiply by the integrating factor gives

$$te^t y' + (t+1)e^t y = te^t \Rightarrow (te^t y)' = te^t \Rightarrow te^t y(t) = \int te^t dt = te^t - \int e^t dt + C = (t-1)e^t + C$$

so

$$y(t) = (1 - \frac{1}{t}) + \frac{Ce^{-t}}{t} = \frac{t-1+Ce^{-t}}{t}$$

Plugging in $y(\ln 2) = 1$ gives

$$1 = \frac{\ln 2 - 1 + C \frac{1}{2}}{\ln 2} \Rightarrow C = 2 \Rightarrow y(t) = \frac{t-1+2e^{-t}}{t}$$

Problem 1.2.28 Multiply by the integrating factor $e^{\frac{2t}{3}}$ to get

$$(e^{\frac{2t}{3}}y)' = (1 - t/2)e^{\frac{2t}{3}}$$

Integrating both side and use integration by part (with $u = 1 - t/2, v = \frac{3}{2}e^{2t/3}$) yields

$$e^{\frac{2t}{3}}y(t) = \int (1-t/2)e^{\frac{2t}{3}} dt = (1-t/2)\frac{3}{2}e^{\frac{2t}{3}} - \int \frac{3}{2}e^{\frac{2t}{3}}(-\frac{1}{2})dt = (1-t/2)\frac{3}{2}e^{\frac{2t}{3}} + \frac{9}{8}e^{\frac{2t}{3}} + C = (21/8 - 3t/4)e^{\frac{2t}{3}} + C$$

Thus, the general solution is

$$y(t) = \frac{21}{8} - \frac{3t}{4} + Ce^{-\frac{2t}{3}}$$

Plugging in initial condition $y(0) = y_0$ yields

$$y_0 = \frac{21}{8} + C \Rightarrow C = y_0 - \frac{21}{8}$$

Thus,

$$y(t) = \frac{21}{8} - \frac{3t}{4} + (y_0 - \frac{21}{8})e^{-\frac{2t}{3}}$$

Suppose the solution touches, but does not cross the t -axis, at time t_0 , then we know that

$$y(t_0) = 0, \text{ and } y'(t_0) = 0$$

Plugging in $t = t_0$ to the ODE

$$y'(t_0) + \frac{2}{3}y(t_0) = 1 - \frac{t_0}{2} \Rightarrow 0 = 1 - \frac{t_0}{2} \Rightarrow t_0 = 2$$

Plugging $t_0 = 2, y(t_0) = 0$ to the solution we derived to find y_0 , that is y_0 satisfies

$$0 = \frac{21}{8} - \frac{3 \cdot 2}{4} + (y_0 - \frac{21}{8})e^{-\frac{2 \cdot 2}{3}}$$

which we can solve for y_0 to get $y_0 = \frac{21}{8} - \frac{3}{4}e^{4/3} = -1.6428$.