Solutions to Math 53 Second Exam —May 13, 2013

1. (20 points) Consider the 2×2 matrix $A = \begin{pmatrix} -7 & 2 \\ -8 & 1 \end{pmatrix}$.

   (a) (a and b together 14 pts) Find the solution $\vec{x}(t)$ of the differential equation $\vec{x}'(t) = A\vec{x}(t)$ with initial condition $\vec{x}(0) = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

   (b) Find the solution $\vec{y}(t)$ of the differential equation $\vec{y}'(t) = A\vec{y}(t)$ with initial condition $\vec{y}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

   (c) (4 pts) Compute the Wronskian determinant of $\vec{x}(t)$ and $\vec{y}(t)$. Do $\vec{x}(t)$ and $\vec{y}(t)$ form a fundamental set of solutions?

   (d) (2 pts) Suppose $\vec{u}(t)$ is the solution of $\vec{u}'(t) = B\vec{u}(t)$ with initial condition $\vec{u}(0) = \begin{pmatrix} 12 \\ -3 \end{pmatrix}$ and $\vec{v}(t)$ is the solution of $\vec{v}'(t) = B\vec{v}(t)$ with initial condition $\vec{v}(0) = \begin{pmatrix} 7 \\ 11 \end{pmatrix}$ for some 2×2 matrix B. Do $\vec{v}(t)$ and $\vec{u}(t)$ form a fundamental set of solutions?

The characteristic polynomial is $(-7-\lambda)(1-\lambda) + 16 = \lambda^2 + 6\lambda + 9 = (\lambda+3)^2$, so we have only one eigenvalue. Since the matrix is not diagonal, we need to find an eigenvector $\vec{v}$ and a generalized eigenvector $\vec{w}$.

To find $\vec{v}$ we solve $\begin{pmatrix} -4 & 2 \\ -8 & 4 \end{pmatrix} \vec{v} = \vec{0}$. Thus $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ works.

To find $\vec{w}$ we solve $\begin{pmatrix} -4 & 2 \\ -8 & 4 \end{pmatrix} \vec{w} = \vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Thus $\vec{w} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$ works.

Hence the general solution is $c_1 e^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left[e^{-3t} t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{-3t} \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \right]$.

(a) We solve $c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ to get $c_1 = 1, c_2 = -10$.

So $\vec{x}(t) = e^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 10 \left[e^{-3t} t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{-3t} \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \right]$.

(b) We solve $c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ to get $c_1 = 2, c_2 = -2$.

So $\vec{y}(t) = 2e^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 6 \left[e^{-3t} t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{-3t} \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \right]$.

(c) The Wronskian at time zero is $\begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} = 7$. Hence $W[\vec{x}, \vec{y}](t) = 7e^{-6t}$. Since the Wronskian is non-zero, $\vec{x}(t)$ and $\vec{y}(t)$ form a fundamental set of solutions.

(d) Yes, the Wronskian at time zero is non-zero, hence it stays non-zero for all time and $\vec{u}(t)$ and $\vec{v}(t)$ form a fundamental set of solutions.

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2. (20 points) Consider the homogeneous system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}.$$

   (a) (10 points) What condition on the eigenvalues of A corresponds to this system having:

      (i) a spiral sink?
      (ii) nodal source?
      (iii) nodal sink?

   In all cases, draw a sample phase portrait, being sure to note the direction of trajectories. In nodal cases, clearly indicate on your picture the eigenvectors and which eigenvector has larger eigenvalue.

   (b) (10 points) Now suppose that

$$A = \begin{pmatrix} a & a \\ a & 2 \end{pmatrix}$$

   For which values of a does the corresponding differential equation have a source, sink or saddle at the origin?

(a) Spiral sink: The eigenvalues should be complex and of the form $a \pm ib$ where $a < 0$.

Nodal source: The eigenvalues should both be real and positive.

Picture: in the case of nodal source, the trajectories should go from $\mathbf{v}_2$ to $\mathbf{v}_1$ when the eigenvalues are $\lambda_1 > \lambda_2$, and their directions should be appropriately consistent.

(b) The characteristic polynomial is $(\lambda - x)(\lambda - 2) - x^2$, so we have to solve

$$\lambda^2 - (x+2)\lambda + (2x - x^2) = 0.$$

The discriminant of this quadratic is $(x+2)^2 - 4(2x - x^2) = 5x^2 - 4x + 4$; this is always positive, so the characteristic polynomial always has two real roots. (Short-cut to this conclusion: the matrix is symmetric.)

Next, the product of the eigenvalues is $2x - x^2 = x(2-x)$. This product is negative for $x > 2$ or $x < 0$; for those x, we get a nodal saddle. For $0 < x < 2$, the product is positive, and the sum of the eigenvalues is $x + 2$; thus, we have a nodal source.

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3. (20 points) Consider the system of equations $\frac{dx}{dt} = 3x - y$, $\frac{dy}{dt} = 5x - 3y$. corresponding to the matrix $\begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix}$.

   (a) (6 points) Find the solution with initial condition $x(0) = 1, y(0) = 3$.

   (b) (14 points) Find the solution to the system $\frac{dx}{dt} = 3x - y + t$, $\frac{dy}{dt} = 5x - 3y$ with the initial condition $x(0) = -1/4, y(0) = 0$.

(a) The matrix is $A = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix}$ with char. poly. $(\lambda - 3)(\lambda + 3) + 5 = \lambda^2 - 4$ i.e. $\lambda = \pm 2$. Say $\lambda_1 = 2$ and $\lambda_2 = -2$.

We find the eigenvectors:

$$\begin{pmatrix} 1 & -1 \\ 5 & -5 \end{pmatrix} v_1 = 0,$$

so we can take $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. and

$$\begin{pmatrix} 5 & -1 \\ 5 & -1 \end{pmatrix} v_2 = 0,$$

so we can take $v_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$. The general solution is

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{-2t}$$

and the initial condition means that

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

we take $c_1 = c_2 = 1/2$.

(b) We use variation of parameters: Let $X = \begin{pmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{pmatrix}$ be the solution matrix and take the solution of the form XU. The equation becomes $\frac{dX}{dt}U + X\frac{dU}{dt} = AXU + \begin{pmatrix} t \\ 0 \end{pmatrix}$, so that

$$X\frac{dU}{dt} = \begin{pmatrix} t \\ 0 \end{pmatrix} \implies \frac{dU}{dt} = X^{-1}\begin{pmatrix} t \\ 0 \end{pmatrix}.$$

Now $\det(X) = 4$, so $X^{-1} = \frac{1}{4}\begin{pmatrix} 5e^{-2t} & -e^{-2t} \\ -e^{2t} & e^{2t} \end{pmatrix}$, and so

$$\frac{dU}{dt} = \frac{1}{4}\begin{pmatrix} 5te^{-2t} \\ -te^{2t} \end{pmatrix}$$

Integrating,

$$U = \begin{pmatrix} \frac{-5}{2}te^{-2t} + \frac{5}{2}\int e^{-2t} \\ -\frac{1}{2}te^{2t} + \int \frac{1}{2}e^{2t}dt \end{pmatrix} = \begin{pmatrix} \frac{-5}{2}te^{-2t} - \frac{5}{4}e^{-2t} + c_1 \\ -\frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + c_2 \end{pmatrix}.$$

Finally, multiplying this by X, we arrive at the solution

$$XU = \frac{1}{4}\begin{pmatrix} -1 - 3t \\ -5t \end{pmatrix} + \text{old general solution},$$

and the initial condition means that the solution is simply $\frac{1}{4}\begin{pmatrix} -1 - 3t \\ -5t \end{pmatrix}$.

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4. (20 points)

(a) (10 points) Write down the solution $\mathbf{v}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ to the system of equations

$$\frac{dx}{dt} = 4x - 5y, \frac{dy}{dt} = 4x \tag{1}$$

with initial values $x(0) = 5, y(0) = -2$, in terms of real-valued functions.

(b) (5 points) Let $\mathbf{v}(t)$ be the solution from (i). Show that $\mathbf{u}(t) = \frac{d}{dt}\mathbf{v}(t)$ is *also* a solution to (1) and evaluate the Wronskian of $\mathbf{v}(t)$ and $\mathbf{u}(t)$ as a function of t.

(c) (5 points) Suppose the system $\frac{dx}{dt} = ax + by, \frac{dy}{dt} = cx + dy$ is known to have complex eigenvalues. Give an interpretation in terms of trajectories for each of the following conditions:

(i) $a + d > 0$;

(ii) $b > 0$.

(a) The corresponding matrix is $A = \begin{pmatrix} 4 & -5 \\ 4 & 0 \end{pmatrix}$, with characteristic polynomial $(4-\lambda)(-\lambda)+20 = 0$, i.e. $\lambda^2 - 4\lambda + 20 = 0$. The roots are $\lambda = 2 \pm 4i$.

The corresponding eigenvector for $\lambda = 2 + 4i$ solves

$$\begin{pmatrix} 2-4i & -5 \\ 4 & -2+4i \end{pmatrix} \mathbf{v} = 0$$

so we can take $\mathbf{v} = \begin{pmatrix} 5 \\ 2-4i \end{pmatrix}$. A fundamental set of solutions is then given by the real and imaginary parts of $\begin{pmatrix} 5 \\ 2-4i \end{pmatrix} e^{2+4i}$:

$$e^{2t} \begin{pmatrix} 5\cos(4t) + 5i\sin(4t) \\ (2-4i)(\cos(4t) + i\sin(4t)) \end{pmatrix} = e^{2t} \begin{pmatrix} 5\cos(4t) + 5i\sin(4t) \\ (2\cos(4t) + 4\sin(4t)) + i(2\sin(4t) - 4\cos(4t)) \end{pmatrix}$$

and so

$$c_1 \begin{pmatrix} 5\cos(4t) \\ 2\cos(4t) + 4\sin(4t) \end{pmatrix} + c_2 \begin{pmatrix} 5\sin(4t) \\ (2\sin(4t) - 4\cos(4t)). \end{pmatrix}$$

is a general solution. To solve the initial value problem we must have

$$c_1 \begin{pmatrix} 5 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}.$$

so that $c_1 = 1$ and $c_2 = 1$; the solution is

$$\begin{pmatrix} 5\cos(4t) + 5\sin(4t) \\ -2\cos(4t) + 6\sin(4t) \end{pmatrix}.$$

(b) We just differentiate the equations: We see that

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = 4\frac{dx}{dt} - 5\frac{dy}{dt}, \quad \frac{d}{dt}\left(\frac{dy}{dt}\right) = 4\frac{dx}{dt}.$$

So $(\frac{dx}{dt}, \frac{dy}{dt})$ is a solution, too. Its initial value is given by $(4\times 5 + 5\times 2, 4\times 5) = (30, 20)$. Thus the Wronskian at time 0 has value $\det \begin{bmatrix} 5 & 30 \\ -2 & 20 \end{bmatrix} = 160$. Thus the Wronskian at time t is $160e^{4t}$.

(c) Having $a + d > 0$ means the sum of the two eigenvalues is positive, i.e. the real part is positive, so it is a spiral sink. Having $b > 0$ means that the direction of trajectory at $x = 0, y = 1$ is in the direction of increasing x, i.e. *clockwise.*

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5. (20 points) Consider an isolated system of two bodies exchanging heat. Denote the temperature of the first body by $T_1(t)$ and the temperature of the second body by $T_2(t)$ (we assume that the temperature does not vary inside each of the two bodies). Then the Newton's law of cooling says that

$$\frac{dT_1}{dt} = k_1(T_2 - T_1)$$
$$\frac{dT_2}{dt} = k_2(T_1 - T_2)$$

(a) (9 pts) Compute the eigenvalues and eigenvectors for the above system of ODEs. Write down the general solution. Describe what happens to this solution when time goes to infinity and explain in a few words what it means for the physical system being modeled.

(b) (3pts) Given $k_1 = 1$ and $k_2 = 2$, draw the phase portrait as well as you can.

(c) (8 pts) Suppose now that the second body is connected to a thermal bath - an external environment that maintains a constant temperature T. This changes the ODE system to

$$\begin{aligned}\frac{dT_1}{dt} &= k_1(T_2 - T_1)\\ \frac{dT_2}{dt} &= k_2(T_1 - T_2) + k_3(T - T_2)\end{aligned}$$

Note that this is now an inhomogeneous system of ODEs.

Make the substitution $T_1 = T + u$ and $T_2 = T + v$ and show that u, v satisfy a homogeneous system of ODEs. Describe what happens to u, v as time goes to infinity, and say in a few words what this means for the physical system being modeled.

(a) The matrix is $A = \begin{pmatrix} -k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix}$, which has characteristic polynomial $(-\lambda - k_1)(-\lambda - k_2) - k_1k_2 = \lambda^2 + \lambda(k_1 + k_2)$. So the eigenvalues are 0 and $-(k_1 + k_2)$. For the 0 eigenvalue we need to solve $\begin{pmatrix} -k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix} \vec{v_1} = 0$. The vector $\vec{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ works. For $-(k_1 + k_2)$ we need to solve $\begin{pmatrix} k_2 & k_1 \\ k_2 & -k_1 \end{pmatrix} \vec{v_2} = 0$. The vector $\vec{v_2} = \begin{pmatrix} k_1 \\ -k_2 \end{pmatrix}$ works. The general solution is $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-(k_1+k_2)t} \begin{pmatrix} k_1 \\ -k_2 \end{pmatrix}$. As t goes to infinity, the second component dies out and the system converges to an equilibrium state where $T_1 = T_2$. Physically, this means that it approaches a state where the temperatures are equal and there is no heat exchange, and does so exponentially fast.

(b) The general solution is $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. The $y = x$ line consists of equilibria solutions, and the phase flow is along lines parallel to $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ toward these equillibria.

(c) We have $\frac{dT_1}{dt} = \frac{du}{dt}$ and $\frac{dT_2}{dt} = \frac{dv}{dt}$ so

$$\begin{aligned}\frac{du}{dt} &= k_1(T_2 - T_1) = k_1(v - u)\\ \frac{dv}{dt} &= k_2(T_1 - T_2) + k_3(T - T_2) = k_2(u - v) - k_3 v\end{aligned}$$

This is a homogeneous system with the matrix $\begin{pmatrix} -k_1 & k_1 \\ k_2 & -k_2 - k_3 \end{pmatrix}$ The characteristic polynomial is $\lambda^2 + (k_1 + k_2 + k_3)\lambda + k_1k_3$ which has discriminant $(k_1 + k_2 + k_3)^2 - 4k_1k_3 = (k_1 - k_3)^2 + k_2^2 + 2k_2(k_1 + k_3) > 0$. Thus it has real roots, which are both negative since their product k_1k_3 is positive and their sum $-(k_1 + k_2 + k_3)$ is negative. Hence the system has a nodal sink. Thus u and v go to zero exponentially fast. Physically, this means that the system moves toward the equillibrium state $T_1 = T_2 = T$ exponentially fast.

Note: We could have also reasoned backwards - having a source of any kind or a saddle would mean temperature going to infinity for one or both bodies; and having oscillations of any kind would mean heat flowing from the first body to the second and then back, which can not happen in the system like that; finally, since zero is not an eigenvalue and we can exclude the repeated eigenvalue case by examining the matrix, we are left with only the nodal sink as a possibility.

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