Solutions to Math 53 First Exam — April 22, 2013

1. (20 points)

   (a) (6 points) Give examples of a first order differential equation, a second order differential equation, and a nonlinear differential equation.

   (b) (7 points) Solve the equation $\frac{dy}{dx} = 2y + 3$ with initial condition $y(0) = 1$.

   (c) (7 points) Solve the equation $\frac{dy}{dx} = \frac{x\sin(x)}{y}$ with initial condition $y(0) = 1$.

(a) $y' = 0$; $y'' = 0$; $yy' = 1$.

(b) Solution 1: $y' = 2(y + \frac{3}{2})$ so $(y(t) + \frac{3}{2}) = (y(0) + \frac{3}{2})e^{2t}$ and $y(t) = 2.5e^{2t} - 1.5$
Solution 2: Rewriting as $y' - 2y = 3$. Integrating factor is e^{-2t}. Solution $y(t)e^{-2t} = 3(\frac{1}{-2}e^{-2t} + C)$, $y(t) = -1.5 + Ce^{2t}$. Initial condition gives $C = 2.5$.

(c) $ydy = x\sin(x)dx$; Integrating by parts we get $\int x\sin(x)dx = -x\cos(x) + \int -\cos(x)dx = -x\cos(x) + \sin(x)$ and so $y^2/2 = -x\cos(x) + \sin(x) + C$. Plugging in the initial value we get $1/2 = C$ so $y^2/2 = -x\cos(x) + \sin(x) + 1/2$, $y = \pm\sqrt{-2x\cos(x) + 2\sin(x) + 1}$ and since $y(0) = +1$ we pick the plus, $y = \sqrt{-2x\cos(x) + 2\sin(x) + 1}$.

2. (20 points) Consider the equation

$$\frac{dM}{dt} = \frac{t}{t^2+1}M + t, \quad M(0) = 2.$$

(a) (12 points) Find the general solution for M by means of an integrating factor, and find $M(10)$.

(b) (8 points) For each of the following modifications the equation or initial conditions, state whether you expect $M(10)$ to *increase* or *decrease* from its value in (ii), and write a sentence or two explaining your reasoning.

- (i) Replacing $\frac{t}{t^2+1}$ by $\left(\frac{t}{t^2+1} + 2e^{-t}\right)$;
- (ii) Replacing the initial condition $M(0) = 2$ by $M(1) = 2$.

(a) $p(t) = -\frac{t}{t^2+1}$ so $\mu(t) = exp \int -\frac{t}{t^2+1}$. To evaluate the integral we put $u = t^2+1$, and $du = 2tdt$, so $\int -\frac{t}{t^2+1} = -\int \frac{1}{2u}du = -\frac{1}{2}\ln|u|+C$. As $u = t^2+1 > 0$ we have $\ln|u| = \ln u$ and we take $C = 0$ to get $\mu(t) = \frac{1}{\sqrt{u}} = \frac{1}{\sqrt{t^2+1}}$; we get $(\frac{M(t)}{\sqrt{t^2+1}})' = \frac{t}{\sqrt{t^2+1}}$. Then $M(t) = \sqrt{t^2+1}\int \frac{t}{\sqrt{t^2+1}}dt$. To evaluate the integral we use the same substitution $u = t^2+1$ and get $\int \frac{1}{2\sqrt{u}}du = \sqrt{u} + C = \sqrt{t^2+1} + C$, so that $M(t) = (t^2+1) + C\sqrt{t^2+1}$.

The initial condition gives $2 = M(0) = 1 + C$ so $C = 1$.

Then $M(10) = 101 + \sqrt{101} \approx 101 + 10.5 = 111.5$.

(b) Solution 1:

We can interpret the equation as describing (for example) the amount of money in a bank, with interest $\frac{t}{t^2+1}$ at time t, starting with 2 dollars at time 0, and we add money at a rate t. (Any similar physical interpretation will lead to similar conclusions.)

Thus, (b)(i) corresponds to increasing the interest rate; we would then expect $M(10)$ to increase from the value in (i). The scenario in (b) (ii) means that we start with 2 dollars at time 1 rather than at time 0 (i.e. it's as if all the money were removed from the account at time 1). Thus, we expect $M(10)$ to decrease from the value in (i).

Solution 2:

(b) (i) Adding the positive term $2e^{-t}$ makes the slopes of the slope field point higher upwards, pushing the solution up as well. The result is that $M(t)$ will be bigger for all $t > 0$ so for $t = 10$ also.

(b) (ii) The slopes are positive, so the old IVP becomes larger than 2 by time 1. After that the two IVP solutions (the old one and the new one) continue on, with the old one above the new one (solutions can not cross because at the crossing point this would violate the uniqueness of solutions), hence the new $M(t)$ is always below the old one for all $t > 1$ and for $t = 10$ as well.

3. (20 points) Let $x(t), y(t)$ be functions of time t that satisfy the equations

$$\frac{dx}{dt} = y, \ \frac{dy}{dt} = -x.$$

Suppose also that $x(0) = 0, y(0) = 5$.

(a) (8 points) The chain rule shows that $\frac{dy}{dx} = -\frac{x}{y}$ (you don't need to prove this). Use separation of variables to show that $y = \sqrt{25 - x^2}$.

(b) (12 points) Substituting the result of (a) into the equation, we find $\frac{dx}{dt} = \sqrt{25 - x^2}$. Solve for x and y as a function of time (the substitution $x = 5\sin(u)$ may help with the integral).

(a) $\frac{dy}{dx} = -\frac{x}{y}$ so $ydy = -xdx$ and $y/2 = -x^2/2 + C$, $y^2 = -x^2 + K$. Plugging in $x(0) = 0$ and $y(0) = 5$ we get $K = 25$ so $y^2 = 25 - x^2$ and $y = \pm\sqrt{25 - x^2}$. Since the solution passes through $(0, 5)$ we must have $y = \sqrt{25 - x^2}$, as wanted.

(b) $\frac{dx}{dt} = \sqrt{25 - x^2}$ is a separable equation. We get $\frac{dx}{\sqrt{25-x^2}} = dt$. We need to integrate $\int \frac{dx}{\sqrt{25-x^2}}$. We make the suggested substitution $x = 5\sin(u)$ for $u \in (-\pi/2, \pi/2)$. Then $dx = 5\cos(u)du$ and $\sqrt{25 - x^2} = |5\cos u| = 5\cos u$. Thus $\int \frac{dx}{\sqrt{25-x^2}} = \int \frac{2\cos(u)du}{5\cos(u)} = u = \arcsin(x/5)$, so $\arcsin(x/5) = t + C$. As $x(0) = 0$ we have $C = 0$, and $x = 5\sin(t)$ for $t \in (-\pi/2, \pi/2)$.

Finally, $y = \frac{dx}{dt} = 5\cos(t)$ for $t \in (-\pi/2, \pi/2)$.

4. (20 points) Consider the equation

$$(y^4 + 3y^2t + 2t^2) + (2y^3t + 2yt^2)\frac{dy}{dt} = 0.$$

(a) (5 points) Determine whether or not this equation is exact.

(b) (5 points) Multiply the equation by $2t$, so that it becomes

$$(2y^4t + 6y^2t^2 + 4t^3) + (4y^3t^2 + 4yt^3)\frac{dy}{dt} = 0.$$

Show it is exact.

(c) (10 points) Suppose that $y(1) = 1$. Solve for y as a function of t.

(a) We compute $(y^4 + 3y^2t + 2t^2)_y = (4y^3 + 6yt) \neq (2y^3 + 4yt) = (2y^3t + 2yt^2)_t$, so it's not exact.

(b) We check $(2y^4t + 6y^2t^2 + 4t^3)_y = 8y^3t + 12yt^2 = (4y^3t^2 + 4yt^3)_t$, so it's exact.

(c) We find $H(t, y) = \int(2y^4t+6y^2t^2+4t^3)dt = t^2y^4+2y^2t^3+t^4+h(y)$; then $H_y = 4y^3t^2+4yt^3+h'(y) = 4y^3t^2 + 4yt^3$, so we take $h(y) = 0$ and $H(t, y) = t^2y^4 + 2y^2t^3 + t^4$. The solutions are given by $t^2y^4 + 2y^2t^3 + t^4 = C$.

Plugging in $y(1) = 1$ we get $1 + 2 + 1 = C$, so $t^2y^4 + 2y^2t^3 + t^4 = 4$. Now we either solve a quadratic equation for y^2 or just notice that the equation is $(ty^2 + t^2)^2 = 4$, $ty^2 + t^2 = \pm 2$. As $y(1) = 1$ we must choose $1 + 1 = +2$, so $ty^2 + t^2 = 2$. Now $y^2 = \frac{2}{t} - t$, $y(t) = \pm\sqrt{\frac{2}{t} - t}$. Again, $y(1) = +1$ forces the choice of the plus sign, so that $y(t) = \sqrt{\frac{2}{t} - t}$.

5. (20 points) Consider the differential equation

$$y' = f(y),$$

where $f(y) = y(y-1)^3(y-2)$.

(a) (4 points) Sketch the graph of $f(y)$.

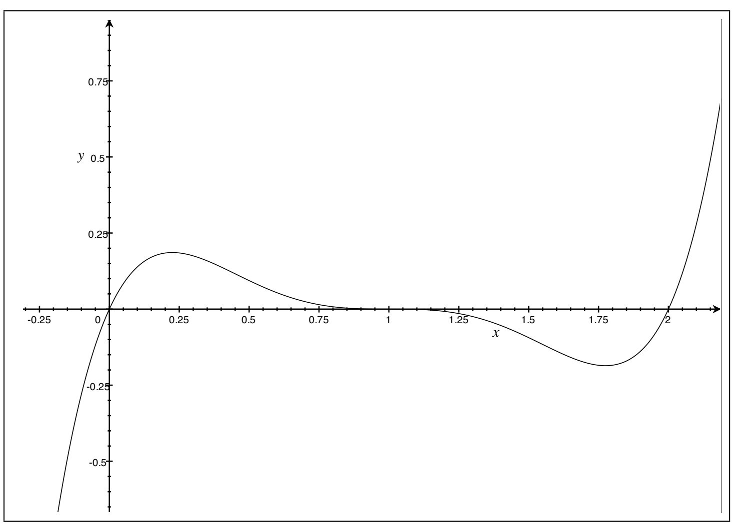
(b) (10 points) Determine all the equilibrium solutions of $y' = f(y)$ and classify them as stable or unstable. Sketch one solution curve of this ODE in each region of the plane cut out by the graphs of the equilibrium solutions.

(c) (6 points) Suppose that $y(0) = 3$. Estimate $y(0.01)$.

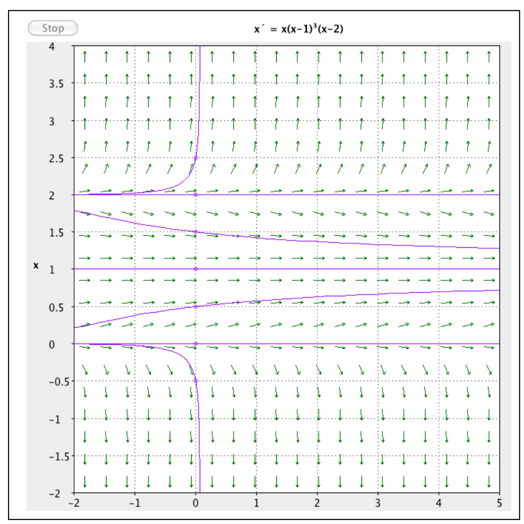
(a) We know the roots are $0, 1$ and 2 and function starts negative and changes sign passing through each root, so we can sketch something approximating the figure below (noticing that 1 is a root of order 3, and so the graph should be fairly flat near it helps make the sketch more accurate, but that is not essential for this problem).

(b) The stationary solutions are $y(t) = 0$ (unstable), $y(t) = 1)$ (stable) and $y(t) = 2$ (unstable). We know that as time goes to infinity, the solutions with initial values of y between 0 and 1 will go up to 1, the ones with initial values of y between 1 and 2 will go down to 1. The solutions starting above 2 will blow up to plus infinity and ones starting below 0 to minus infinity (since the equation is of 5th power they will do so in finite time). Similarly we can find what happens as time goes to minus infinity. A graph is given below.

(c) We use the Euler method with step $h = 0.01$. The slope at $(0, 3)$ is $f(3) = 3 \cdot 2^3 \cdot 1 = 24$. Then $y(0.01) \approx 3 + 24 \cdot 0.01 = 3.24$. (Note: The real answer is about 3.38. The Euler method with $h = 0.005$ gives ≈ 3.28 and with $h = 0.001$ gives ≈ 3.36.)



Graph of $f(y) = y(y-1)^3(y-2)$.



Solution curves of $y' = y(y-1)^3(y-2)$.