All problems are twenty points. This is a closed-book exam with no calculators.

Problem 1.

(a; 6 points) Give an example of a differential equation that is linear and separable, an equation that is linear and *not* separable, and a third order differential equation.

(b; 7 points) Solve the equation $\frac{dy}{dx} = -3y + 5$ with initial condition $y(0) = 1$.

(c; 7 points) Solve the equation $\frac{dy}{dx} = xy + 2y$ with initial condition $y(0) = 1$.

Solution: For (i), $y' = y$ is linear and separable; $y' = xy + 1$ is linear and not separable, and $y''' = y$ is of third order.

For (ii), we may separate: $\frac{dy}{-3y+5} = dx$. Integrating gives $\frac{-1}{3}\log(|-3y+5|) = x + C$, i.e. $(3y-5) = Ke^{-3x}$ where $K = \pm e^{-3C}$. The initial condition means that $K = -2$, so that $3y - 5 = -2e^{-3x}$. Thus the solution is

$$y = \frac{5 - 2e^{-3x}}{3}.$$

For (iii) we may again separate: $\frac{dy}{y} = (x+2)dx$. Integrating gives $\log(|y|) = x^2/2 + 2x + C$, so that $y = Ke^{x^2/2+2x}$ (with $K = \pm e^C$). The initial condition means that $K = 1$, so

$$y = e^{x^2/2+2x}.$$

Problem 2.
Consider the equation

$$y' = \frac{2y}{x} + x^3 e^{-x}.$$

(a; 12 points) Solve the initial value problem $y(1) = 0$ and compute $y(10)$.

(b; 8 points) For each of the following modifications the equation or initial conditions, state whether you expect $y(10)$ to *increase* or *decrease* from its value in (ii), and write a sentence or two explaining your reasoning.

(i) Replacing $\frac{2y}{x}$ by $\frac{4y}{x}$;

(ii) Replacing $y(1) = 0$ by $y(5) = 0$.

Solution: Write the equation as $y' - \frac{2y}{x} = x^3 e^{-x}$. We introduce the integrating factor $e^{\int -\frac{2}{x}dx} = e^{-2\log(x)} = x^{-2}$. The equation becomes

$$x^{-2}y' - \frac{2y}{x^3} = xe^{-x}.$$

The left-hand side is now $\frac{d}{dx}\left(x^{-2}y\right)$, and integrating gives

$$x^{-2}y = \int xe^{-x} = -xe^{-x} + \int e^{-x} = -xe^{-x} - e^{-x} + C,$$

where we used integration by parts. Thus,

$$y = Cx^2 - x^3e^{-x} - x^2e^{-x}.$$

From the initial condition $y(1) = 0$, we see that $C = 2/e$, so the solution is

$$y(x) = (2/e)x^2 - x^3e^{-x} - x^2e^{-x},$$

and $y(10) = \frac{200}{e} - 1100e^{-10}$.

For (b): We can interpret the equation as describing (for example) the amount of money in a bank account with interest $2/x$ at time x, starting with no money at time 1, and we add money at a rate x^3e^{-x}. (Any similar "physical interpretation" will lead to similar conclusions.)

Thus, (b)(i) corresponds to increasing the interest rate; we would then expect $y(10)$ to increase from the value in (i).

(b)(ii) means that we start with no money at time 5 rather than at time 1 – i.e., it's as if all the money were removed from the account at time 5 – thus, we expect $y(10)$ to decrease from the value in (i).

Problem 3. Suppose a function y satisfies

$$\frac{d^2y}{dx^2} = y^{-3}.$$

Write $v = \frac{dy}{dx}$. Using the chain rule, the equation becomes $\frac{dv}{dy}v = y^{-3}$ (you do not need to prove this).

(a; 8 points) Solve this equation to show that $v = \pm\sqrt{(C - \frac{1}{y^2})}$.

(b; 12 points) Assume that we are in the positive square root case from (a), so that $\frac{dy}{dx} = \sqrt{(C - \frac{1}{y^2})}$. Now find the general solution for y as a function of x.

Solutions:

We separate variables to get $vdv = y^{-3}dy$. Integrating and multiplying by 2, we get $v^2 = -y^{-2} + C$ so that $v = \pm\sqrt{C - 1/y^2}$.

For part (b) we separate variables again:

$$\frac{dy}{\sqrt{(C - 1/y^2)}} = dx$$

Integrating, and multiply top and bottom by y on the left:

$$\int \frac{ydy}{\sqrt{(Cy^2 - 1)}} = (x + B)$$

where B is another constant of integration. Now make the substitution $u = Cy^2 - 1$; the left-hand side becomes

$$\frac{1}{2C}\int \frac{du}{\sqrt{u}} = (x + B)$$

so that

$$\frac{1}{C}\sqrt{u} = (x + B)$$

or (squaring and multiplying by C)

$$u = C^2(x + B)^2.$$

Finally, solving for y using $u = Cy^2 - 1$, we get

$$y^2 = \frac{u + 1}{C} = \frac{C^2(x + B)^2 + 1}{C}$$

and finally

$$y = \sqrt{\frac{C^2(x + B)^2 + 1}{C}}$$

Problem 4. Consider the equation

$$(4t^2 + 3ty^2 + 2y) + (2t^2y + t)\frac{dy}{dt} = 0.$$

(a; 5 points) Determine whether or not this equation is exact.

(b; 5 points) Multiply the equation by t, so that it becomes

$$(4t^3 + 3t^2y^2 + 2ty) + (2t^3y + t^2)\frac{dy}{dt} = 0.$$

Show it is exact.

(c; 10 points) Suppose that $y(1) = 1$. Solve for y as a function of t.

Solution. (a) Write $M = (4t^2 + 3ty^2 + 2y)$ and $N = (2t^2y + t)$. To be exact we need to check whether $M_y = N_t$. But $M_y = 6ty + 2$ and $N_t = 4ty + 1$. Since these are not equal, the equation is not exact.

(b) Now write $M = (4t^3+3t^2y^2+2ty)$ and $N = (2t^3y+t^2)$. Then $M_y = 6t^2y+2t$ and $N_t = 6t^2y + 2t$, so the equation is exact.

(c) We need to find a function Ψ with $\Psi_t = M$ and $\Psi_y = N$, where M and N are as defined in (b).

Integrating M with respect to t, we get $\Psi = t^4+t^3y^2+t^2y+A(y)$; differentiating with respect to y we get $\Psi_y = 2yt^3 + t^2 + A'(y)$. Setting this equal to N, we see that $A'(y) = 0$, i.e. A is a constant. So the solution is

$$t^4 + t^3y^2 + t^2y = \text{constant}.$$

Given that $y(1) = 1$, the constant must be equal to 3. So the solution is

$$t^3y^2 + t^2y + (t^4 - 3) = 0.$$

Finally, by the quadratic formula, we get $y = \frac{-t^2 \pm \sqrt{t^4 - 4t^3(t^4-3)}}{2t^3}$. Substituting $t = 1$, we see that we need to take the + sign in the square root in order that $y(1) = 1$, so

$$y = \frac{-t^2 + \sqrt{12t^3 + t^4 - 4t^7}}{2t^3}.$$

Problem 5. Consider the equation

$$\frac{dy}{dx} = f(y)$$

where $f(y) = \frac{(2-y)(1-y)}{y}$, and we will only consider solutions where $y > 0$.

(a; 4 points) Sketch the graph of $f(y)$.

(b; 10 points) Determine all the equilibrium solutions of $y' = f(y)$ and classify them as stable or unstable. Sketch one solution curve of this ODE in each region of the part of the (y, t) plane where $y > 0$.

(c; 6 points) Suppose that $y(0) = 3$. Estimate $y(0.3)$.

Solution to part (c): Since $y'(0) = f(3) = \frac{2}{3}$, we may estimate $y(0.1) \approx y(0) + \frac{2}{3} \times 0.3 = 3.2$. (This is one step of Euler's method, that is to say, simply approximating y by its tangent line near $x = 0$.)