All problems are twenty points. This is a closed-book exam with no calculators.

Problem 1.

(a; 6 points) Give an example of a differential equation that is linear and separable, an equation that is linear and *not* separable, and a third order differential equation.

(b; 7 points) Solve the equation $\frac{dy}{dx} = -3y + 5$ with initial condition $y(0) = 1$.

(c; 7 points) Solve the equation $\frac{dy}{dx} = xy + 2y$ with initial condition $y(0) = 1$.

Problem 2.

Consider the equation

$$y' = \frac{2y}{x} + x^3 e^{-x}.$$

(a; 12 points) Solve the initial value problem $y(1) = 0$ and compute $y(10)$.

(b; 8 points) For each of the following modifications the equation or initial conditions, state whether you expect $y(10)$) to *increase* or *decrease* from its value in (a), and write a sentence or two explaining your reasoning.

(i) Replacing $\frac{2y}{x}$ by $\frac{4y}{x}$;

(ii) Replacing $y(1) = 0$ by $y(5) = 0$.

Problem 3. Suppose a function y satisfies

$$\frac{d^2y}{dx^2} = y^{-3}.$$

Write $v = \frac{dy}{dx}$. Using the chain rule, the equation becomes $\frac{dv}{dy} v = y^{-3}$ (you do not need to prove this).

(a; 8 points) Solve this equation to show that $v = \pm\sqrt{(C - \frac{1}{y^2})}$.

(b; 12 points) Assume that we are in the positive square root case from (a), so that $\frac{dy}{dx} = \sqrt{(C - \frac{1}{y^2})}$. Now find the general solution for y as a function of x.

Problem 4. Consider the equation

$$(4t^2 + 3ty^2 + 2y) + (2t^2y + t)\frac{dy}{dt} = 0.$$

(a; 5 points) Determine whether or not this equation is exact.

(b; 5 points) Multiply the equation by t, so that it becomes

$$(4t^3 + 3t^2y^2 + 2ty) + (2t^3y + t^2)\frac{dy}{dt} = 0.$$

Show it is exact.

(c; 10 points) Suppose that $y(1) = 1$. Solve for y as a function of t.

Problem 5. Consider the equation

$$\frac{dy}{dx} = f(y)$$

where $f(y) = \frac{(2-y)(1-y)}{y}$, and we will only consider solutions where $y > 0$.

(a; 4 points) Sketch the graph of $f(y)$.

(b; 10 points) Determine all the equilibrium solutions of $y' = f(y)$ and classify them as stable or unstable. Sketch one solution curve of this ODE in each region of the part of the (y, t) plane where $y > 0$.

(c; 6 points) Suppose that $y(0) = 3$. Estimate $y(0.2)$.