Solutions to Math 53 Practice Second Midterm

1. (20 points)

   (a) (10 points) Write down the general solution to the system of equations $\frac{dx}{dt} = x+y$, $\frac{dy}{dt} = -13x-3y$ in terms of real-valued functions.

   (b) (5 points) The trajectories of this equation in the (x, y)-plane rotate around the origin. Is this rotation clockwise or counterclockwise? If we changed the first equation to $\frac{dx}{dt} = -x+y$, would it change the direction of rotation?

   (c) (5 points) Suppose $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are two solutions to this equation. Define the Wronskian determinant $W(t)$ of these two solutions. If $W(0) = 1$, what is $W(10)$? *Note: the material for this part will be covered in the May 8 and May 10 lectures.*

(a) The corresponding matrix is $A = \begin{pmatrix} 1 & 1 \\ -13 & -3 \end{pmatrix}$, with characteristic polynomial $(1-\lambda)(-3-\lambda)+13 = 0$, so that $\lambda^2+2\lambda+10 = 0$, so that $\lambda = -1 \pm 3i$. The corresponding eigenvector $\mathbf{v}$ to $\lambda_1 = -1+3i$ must be a solution to

$$\begin{pmatrix} 2-3i & 1 \\ -13 & -2-3i \end{pmatrix}$$

so we can take $\mathbf{v} = \begin{pmatrix} 1 \\ 3i-2 \end{pmatrix}$ Now we just need to compute the real and imaginary parts of

$$\mathbf{v}e^{\lambda_1 t} = e^{-t}\begin{pmatrix} e^{3it} \\ (3i-2)e^{3it} \end{pmatrix} = e^{-t}\begin{pmatrix} \cos(3t) + i\sin(3t) \\ -2\cos(3t) - 3\sin(3t) + i(3\cos(3t) - 2\sin(3t)) \end{pmatrix}.$$

The general solution is thus expressed as

$$c_1 e^{-t}\begin{pmatrix} \cos(3t) \\ -2\cos(3t) - 3\sin(3t) \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin(3t) \\ (3\cos(3t) - 2\sin(3t)) \end{pmatrix}.$$

(b) We can examine the direction field. For example, when $(x, y) = (1, 0)$, we have $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = -13$. This corresponds to clockwise rotation (draw a picture to see this.) If we replace the first equation by $\frac{dx}{dt} = -x+y$, we still get clockwise rotation (by a similar argument).

(c) The Wronskian is defined as the determinant of $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$, i.e. $W(t) = x_1y_2 - y_1x_2$.

It is a general fact that for the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the Wronskian will satisfy $W' = (a+d)W$ and thus is of the form $W(t) = Ce^{(a+d)t}$. So, if $W(0) = 1$, then $W(t) = e^{-2t}$ so $W(10) = e^{-20}$.

2. (20 points)

   Consider the 2×2 matrix $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$.

   (a) (14 points, together with (b)) Find the solution $\vec{x}(t)$ of the differential equation $\vec{x}'(t) = A\vec{x}(t)$ with initial condition $\vec{x}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

   (b) (14 points, together with (a)) Find the solution $\vec{y}(t)$ of the differential equation $\vec{y}'(t) = A\vec{y}(t)$ with initial condition $\vec{y}(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

(c) (6 points) Compute the Wronskian determinant of $\vec{x}(t)$ and $\vec{y}(t)$. Do $x(t)$ and $y(t)$ form a fundamental set of solutions?

The matrix has characteristic polynomial $(3-\lambda)(1-\lambda)+1 = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$, so it has one eigenvalue 2. As it is not diagonal, we have only one eigenvector $\vec{v}$ which we find by solving $(A-\lambda I)\vec{v} = \begin{pmatrix} 3-2 & 1 \\ -1 & 1-2 \end{pmatrix}\vec{v} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\vec{v} = \vec{0}$. The vector $\vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ works.

We then find a generalized eigenvector $\vec{w}$ by solving $(A-\lambda I)\vec{w} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\vec{w} = \vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

The vector $\vec{w} = \begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix}$ works.

So the general solution is $c_1 e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\left[te^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{2t}\begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix}\right]$.

(a) $\vec{x}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ translates to $c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ which we can solve to give $c_2 = -1$ and $c_1 = 3/2$, so that the solution is $-e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + 3/2\left[te^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{2t}\begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix}\right]$.

(b) $\vec{y}(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ translates to $c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ which we can solve to give $c_1 = 0$ and $c_2 = -4$, so that the solution is $-4\left[te^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{2t}\begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix}\right]$.

(c) This can be computed directly, but that is quite tedious. On the other hand, $W(t) = e^{trA}W(0)$, so we have $W(t) = e^{4t}\begin{vmatrix} -1 & 2 \\ 2 & 2 \end{vmatrix} = -6e^{4t}$. The solutions $x(t)$ and $y(t)$ do from a fundamental set.

3. (20 points) Consider the homogeneous system

$$\frac{d}{dt}\mathbf{v} = A\mathbf{v}$$

defined by a 2×2 matrix A.

(a) (5 points) What condition on the eigenvalues of A corresponds to the origin being a nodal source? Draw a sample phase portrait, with arrows marking the direction of trajectories.

(b) (15 points) Now suppose that

$$A = \begin{pmatrix} x & 1 \\ 1-x & 2 \end{pmatrix}.$$

For what values of x does the corresponding differential equation have a source, sink, or saddle at the origin?

(a) Nodal source: the eigenvalues are real and positive. See Figure 3.3.2 of the book. Your picture should be sure to note the direction of several sample trajectories.

(b) The characteristic polynomial is

$$(x-\lambda)(2-\lambda) - (1-x) = 0,$$

that is

$$\lambda^2 - (x+2)\lambda + (3x-1) = 0.$$

The solutions are

$$\lambda_1, \lambda_2 = \frac{(x+2) - \sqrt{(x+2)^2 - 4(3x-1)}}{2}.$$

Note that $\lambda_1 + \lambda_2 = (x+2)$ and $\lambda_1\lambda_2 = 3x-1$. (You don't need to use the quadratic formula to see this – instead, equate coefficients in $(x-\lambda_1)(x-\lambda_2) = \lambda^2 - (x+2)\lambda + (3x-1)$.)

Firstly, the eigenvalues will be complex when

$$(x+2)^2 < 4(3x-1)$$

i.e. $x^2 - 8x + 8 < 0$. The roots of $x^2 - 8x + 8 = 0$ are $a = 4 + 2\sqrt{2}$ and $b = 4 - 2\sqrt{2}$. For $a < x < b$, the eigenvalues will then be complex. Note that $a \approx 6.8$ and $b \approx 1.2$.

Also, the real part of the eigenvalues will be given by $\frac{x+2}{2} > 0$, so the origin will be a spiral source.

Split into cases:

- $x < 1/3$. We've seen the eigenvalues are real; also, $\lambda_1\lambda_2 = 3x - 1 < 0$, so the eigenvalues are real and of opposite sign $\implies$ SADDLE.
- $1/3 < x < b$. Here we've seen the eigenvalues are real; also, $\lambda_1\lambda_2 = 3x - 1 > 0$, so the eigenvalues must have the same sign. Also, because $\lambda_1 + \lambda_2 = x + 2 > 0$, the eigenvalues must both be positive $\implies$ SOURCE.
- $b < x < a$. The eigenvalues are complex, with positive real part $\implies$ (spiral) SOURCE.
- $x > a$: Here we've seen the eigenvalues are real; also, $\lambda_1\lambda_2 = 3x - 1 > 0$ and $\lambda_1 + \lambda_2 = x + 2 > 0$ so the eigenvalues are of the same sign and positive $\implies$ SOURCE.

4. (20 points) Consider the system of equations $\frac{dx}{dt} = 3x + 4y$, $\frac{dy}{dt} = y + 3x$.
   (a) (10 points) Find the solution with initial condition $x(0) = 4, y(0) = 0$.
   (b) (10 points) Find the solution to the system $\frac{dx}{dt} = 3x + 4y + e^{5t}$, $\frac{dy}{dt} = x + 3y$ with the same initial condition. (*Note to students: The solution method for this will be covered in the May 8 and May 10 lectures.*)

(a) **Note:** There was a typo in the original question – the equation should have been $\frac{dy}{dt} = x + 3y$, matching the equation written in (b). If you worked the original version of part (a), you would have gotten eigenvalues $2 \pm \sqrt{13}$ instead. The solution that follows is for the system $\frac{dx}{dt} = 3x + 4y$, $\frac{dy}{dt} = x + 3y$.

The matrix is $A = \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix}$ with char. poly. $(\lambda-3)(\lambda-3) - 4 = \lambda^2 - 6\lambda + 5 = (\lambda-5)(\lambda-1)$, i.e. $\lambda_1 = 5, \lambda_2 = 1$.

We find the eigenvectors:

$$\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} v_1 = 0,$$

so we can take $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. and

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} v_2 = 0,$$

so we can take $v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. The general solution is

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t}$$

and the initial condition means that

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

so the solution corresponds to $c_1 = c_2 = 1$, i.e.

$$x = 2e^{5t} + 2e^{t}, y = e^{5t} - e^{t}.$$

(b) We use variation of parameters: Let $X = \begin{pmatrix} 2e^{5t} & 2e^{t} \\ e^{5t} & -e^{t} \end{pmatrix}$ be the solution matrix and take the solution of the form XU. The equation becomes $\frac{dX}{dt}U + X\frac{dU}{dt} = AXU + \begin{pmatrix} e^{5t} \\ 0 \end{pmatrix}$, so that

$$X\frac{dU}{dt} = \begin{pmatrix} e^{5t} \\ 0 \end{pmatrix}.$$

Write $U = \begin{pmatrix} f \\ g \end{pmatrix}$. We get $f' = 1/4, g' = e^{4t}/4$. Integrating, a solution is $u = t/4, v = e^{4t}/16$. The solution is therefore

$$x = (t/2)e^{5t} + e^{5t}/8 + 2c_1e^{5t} + 2c_2e^{t}, y = (t/4)e^{5t} - e^{5t}/16 + c_1e^{5t} - c_2e^{t},$$

For the initial condition we must have

$$\frac{1}{8} + 2c_1 + 2c_2 = 4, \quad -1/16 + c_1 - c_2 = 0.$$

so that $c_1 = 1, c_2 = 15/16$, so finally

$$x = (t/2)e^{5t} + \frac{17}{8}e^{5t} + \frac{15}{8}e^{t}, y = (t/4)e^{5t} + \frac{15}{16}e^{5t} - \frac{15}{16}e^{t},$$

5. (20 points) According to Hooke's law, a unit mass oscillating on a spring without friction is described by a second order differential equation $x'' = -kx$, where k is a spring constant, a positive real number. We can turn this into a second order system by putting $y = x'$ so that $y' = x''$ and so that

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -kx$$

(a) (4 points) State the type of critical point the system has, and describe in words the corresponding motion of the spring.

(b) (4 points) Adding friction modifies the equation to be $x'' = -kx - cx'$, where c is a friction coefficient, a positive real number. The system becomes (still, with $y = x'$)

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -kx - cy$$

Suppose the friction coefficient c is very small (compared to 1 and k). State the type of critical point the system has, and describe in words the corresponding motion of the spring.

(c) (4 points) We continue to look at a oscillating mass on a spring with friction, as in part b, but now we suppose the friction coefficient c is very large. State the type of critical point the system has, and describe in words the corresponding motion of the spring.

(d) (8 points) For what value of c does the transition between the two types of critical pints in parts b) and c) happens? Compute and give your answer as a formula for the value of c as a function of k. What is thetype of singularity for that transition value of c?

(a) The spring is oscillating forever, with constant amplitude. This corresponds to a *center* critical point.

(b) The spring will oscillate with decreasing amplitude, moving towards the equilibrium solution of zero displacement and zero velocity. This is a *spiral sink*.

(c) For very large friction, the system will not oscillate, but rather just decrease displacement and velocity towards the equilibrium. This is a *nodal sink*.

(d) We consider the characteristic polynomial of the system $-\lambda(-\lambda - c) + k = \lambda^2 + c\lambda + k$. The determinant is $c^2 - 4k$, which when we increase c grows from $-4k$ to positive infinity. The transition happens when it is equal to zero (we switch from complex to real eigenvalues), that is for $c = 2\sqrt{k}$. For that c we have an *improper nodal sink*.

(This page intentionally left blank for your work.)