Math 53 Practice Second Midterm

Name: ______________________________ **SUID#:** ____________

Circle your section:		
Ulrik Buchholtz 11 (2:15-3:05 PM) 5 (11:00-11:50 PM)	Xiaodong Li 8 (11:00-11:50 AM) 9 (10:00-10:50 AM)	Saran Ahuja 3 (1:15 - 2:05 PM) 10 (10:00-10:50 AM)

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so. You may use any result proved in class or the text, but be sure to clearly state the result before using it, and to verify that all hypotheses are satisfied.
- Please check that your copy of this exam contains 3 numbered pages and is correctly stapled.
- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- **You have 2 hours.** Your organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.
- Paper not provided by teaching staff is prohibited. If you need extra room for your answers, use the back side of a page or one of the extra sheets provided in this packet, and clearly indicate that your answer continues there. Do not unstaple or detach pages from this exam.
- It is your responsibility to arrange to pick up your graded exam paper from your section leader in a timely manner. You have only until **Thursday, May 9**, to resubmit your exam for any regrade considerations; consult your section leader about the exact details of the submission process.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________________

The following boxes are strictly for grading purposes. Please do not mark.

Question:	1	2	3	4	5	**Total**
Points:	20	20	20	20	20	100
Score:						

1. (20 points)

   (a) (10 points) Write down the general solution to the system of equations $\frac{dx}{dt} = x + y$, $\frac{dy}{dt} = -13x - 3y$ in terms of real-valued functions.

   (b) (5 points) The trajectories of this equation in the (x, y)-plane rotate around the origin. Is this rotation clockwise or counterclockwise? If we changed the first equation to $\frac{dx}{dt} = -x + y$, would it change the direction of rotation?

   (c) (5 points) Suppose $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are two solutions to this equation. Define the Wronskian determinant $W(t)$ of these two solutions. If $W(0) = 1$, what is $W(10)$? *Note: the material for this part will be covered in the May 8 and May 10 lectures.*

2. (20 points)

   Consider the 2×2 matrix $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$.

   (a) (14 points, together with (b)) Find the solution $\vec{x}(t)$ of the differential equation $\vec{x}'(t) = A\vec{x}(t)$ with initial condition $\vec{x}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

   (b) (14 points, together with (a)) Find the solution $\vec{y}(t)$ of the differential equation $\vec{y}'(t) = A\vec{y}(t)$ with initial condition $\vec{y}(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

   (c) (6 points) Compute the Wronskian determinant of $\vec{x}(t)$ and $\vec{y}(t)$. Do $x(t)$ and $y(t)$ form a fundamental set of solutions?

3. (20 points) Consider the homogeneous system

$$\frac{d}{dt}\mathbf{v} = A\mathbf{v}$$

   defined by a 2×2 matrix A.

   (a) (5 points) What condition on the eigenvalues of A corresponds to the origin being a nodal source? Draw a sample phase portrait, with arrows marking the direction of trajectories.

   (b) (15 points) Now suppose that

$$A = \begin{pmatrix} x & 1 \\ 1 - x & 2 \end{pmatrix}.$$

   For what values of x does the corresponding differential equation have a source, sink, or saddle at the origin?

4. (20 points) Consider the system of equations $\frac{dx}{dt} = 3x + 4y$, $\frac{dy}{dt} = y + 3x$.

   (a) (10 points) Find the solution with initial condition $x(0) = 4, y(0) = 0$.

   (b) (10 points) Find the solution to the system $\frac{dx}{dt} = 3x + 4y + e^{5t}$, $\frac{dy}{dt} = x + 3y$ with the same initial condition. (*Note to students: The solution method for this will be covered in the May 8 and May 10 lectures.*)

5. (20 points) According to Hooke's law, a unit mass oscillating on a spring without friction is described by a second order differential equation $x'' = -kx$, where k is a spring constant, a positive real number. We can turn this into a second order system by putting $y = x'$ so that $y' = x''$ and so that

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -kx$$

(a) (4 points) State the type of critical point the system has, and describe in words the corresponding motion of the spring.

(b) (4 points) Adding friction modifies the equation to be $x'' = -kx - cx'$, where c is a friction coefficient, a positive real number. The system becomes (still, with $y = x'$)

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -kx - cy$$

Suppose the friction coefficient c is very small (compared to 1 and k). State the type of critical point the system has, and describe in words the corresponding motion of the spring.

(c) (4 points) We continue to look at a oscillating mass on a spring with friction, as in part b, but now we suppose the friction coefficient c is very large. State the type of critical point the system has, and describe in words the corresponding motion of the spring.

(d) (8 points) For what value of c does the transition between the two types of critical pints in parts b) and c) happens? Compute and give your answer as a formula for the value of c as a function of k. What is thetype of singularity for that transition value of c?

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