5.1.

11. As $t \geq 0$, $\left|\frac{1}{1+t}\right| \leq 1 = 1 \cdot e^{0 \cdot t}$.

Therefore, $\frac{1}{1+t}$ is of exponential order. $M=0$, $K=1$, $a=0$

15. $f(t) = \begin{cases} 0 & 0 \leq t \leq 1 \\ 1 & 1 < t \leq 2 \\ 0 & 2 < t \end{cases}$

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_1^2 e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_1^2 = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

18. $f(t) = \cosh(bt) = \dfrac{e^{bt} + e^{-bt}}{2}$

$$f(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty \frac{e^{(b-s)t} + e^{-(b+s)t}}{2}\, dt$$

$$= \frac{1}{2(b-s)} e^{(b-s)t}\Big|_0^\infty + \frac{1}{2(-b-s)} e^{-(b+s)t}\Big|_0^\infty$$

$$= \frac{1}{2(b+s)} + \frac{1}{2(s-b)} = \frac{s}{s^2-b^2} \quad (s > |b|)$$

20. $f(t) = e^{at}\cosh(bt) = e^{at}\dfrac{e^{bt}+e^{-bt}}{2}$

$$= \frac{e^{(a+b)t}+e^{(a-b)t}}{2}$$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st}\,\frac{e^{(a+b)t}+e^{(a-b)t}}{2}\,dt$$

$$= \frac{1}{2}\left(\int_0^\infty e^{(a+b-s)t}\,dt + \int_0^\infty e^{(a-b-s)t}\,dt\right)$$

$$= \frac{1}{2}\left(\frac{1}{a+b-s} e^{(a+b-s)t}\Big|_0^\infty + \frac{1}{a-b-s} e^{(a-b-s)t}\Big|_0^\infty\right)$$

$$\overset{s-a>|b|}{=} \frac{1}{2}\left(\frac{1}{s-a-b} + \frac{1}{s+b-a}\right) = \frac{s-a}{(s-a)^2-b^2}$$

5.2

11.(a) Let $g_1(t) = \int_0^t f(t_1)\,dt_1$, we have $g_1'(t) = f(t)$.

and $g_1(0) = 0$.

Then $\mathcal{L}\{g_1'(t)\} = s\mathcal{L}\{g_1(s)\} - g_1(0)$

$$= s\mathcal{L}\{\int_0^t f(\tau)\,d\tau\}$$

By the fact $\mathcal{L}\{g_1'(t)\} = \mathcal{L}\{f(t)\} = F(s)$,

we have $\mathcal{L}\{\int_0^t f(\tau)\,d\tau\} = \frac{F(s)}{s}$.

(b). Let $g(t) = \int_0^t \int_0^{t_{n-1}} \cdots \int_0^{t_2} f(t_1)\,dt_1 \cdots dt_{n-1}$.

Then $\mathcal{L}\{\int_0^t \int_0^{t_n} \cdots \int_0^{t_2} f(t_1)\,dt_1 \cdots dt_n\}$

$$= \mathcal{L}\{\int_0^t g(t_n)\,dt_n\} = \frac{1}{s}\mathcal{L}\{g(t)\}$$

$$= \frac{1}{s}\mathcal{L}\{\int_0^t \int_0^{t_{n-1}} \cdots \int_0^{t_2} f(t_1)\,dt_1 \cdots dt_{n-1}\}$$

By using this identity iteratively, we have

$$\mathcal{L}\{\int_0^t \int_0^{t_n} \cdots \int_0^{t_2} f(t_1)\,dt_1 \cdots dt_n\} = \frac{1}{s}\mathcal{L}\{\int_0^t \int_0^{t_{n-1}} \cdots \int_0^{t_2} f(t_1)\,dt_1 \cdots dt_{n-1}\}$$

$$= \frac{1}{s^2}\mathcal{L}(\int_0^t \int_0^{t_{n-2}} \cdots \int_0^{t_2} f(t_1)\,dt_1 \cdots dt_{n-2}\} = \cdots = \frac{1}{s^n}F(s).$$

20. $y''' + y'' + y' + y = 0$

$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2$

Taking the Laplace transform, we have

$$\left(s^3 Y - s^2 y(0) - s y'(0) - y''(0)\right)$$
$$+ \left(s^2 Y - s y(0) - y'(0)\right) + \left(sY - y(0)\right) + Y = 0.$$

Then we have

$$(s^3 + s^2 + s + 1) Y = s^2 + s - 1. \qquad Y = \frac{s^2 + s - 1}{s^3 + s^2 + s + 1}.$$

22. $L q'' + R q' + \frac{1}{C} q = e(t)$

By taking the Laplace transform, we have

$$L\left(s^2 Q(s) - s q(0) - q'(0)\right) + R\left(sQ(s) - q(0)\right) + \frac{1}{C} Q(s) = E(s)$$

Then $\left(Ls^2 + Rs + \frac{1}{C}\right) Q(s) = E(s) + L q'(0) + (Ls + R) q(0)$

By the definition $i = q'$, we have

$$Q(s) = \frac{(Ls + R) q(0) + L i(0) + E(s)}{Ls^2 + Rs + \frac{1}{C}}$$

By $I(s) = sQ(s) - q(0)$, we have

$$I(s) = \frac{Ls\, i(0) + sE(s) - q(0)/C}{Ls^2 + Rs + \frac{1}{C}}$$

23. ~~Suff~~ Suppose $f(t) = \begin{cases} 1 & 0 \le t < \pi \\ 0 & \pi \le t < \infty \end{cases}$

Then $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\pi e^{-st}\,dt$

$$= -\frac{1}{s} e^{-st} \Big|_0^1 = \frac{1}{s} - \frac{1}{s} e^{-\pi s}$$

The Laplace transform of the equation

$$y'' + 4y = f(t) \quad \text{is}$$

$$\left(s^2 Y - s y(0) - y'(0)\right) + 4Y = \frac{1}{s} - \frac{1}{s} e^{-\pi s}.$$

By $y(0) = 1$, $y'(0) = 0$, we have

$$Y = \frac{1}{4+s^2}\left(s + \frac{1}{s} \cancel{0} - \frac{1}{s} e^{-\pi s}\right)$$

$$= \frac{s^2+1}{s(4+s^2)} - \frac{e^{-\pi s}}{s(4+s^2)}$$

5.3

6. $$\frac{a_1}{s} + \frac{a_2}{s^2} + \frac{b}{s+3} + \frac{c}{s+1}$$

$$= \frac{1}{s^2(s+1)(s+3)}\Big[s(s+3)(s+1)a_1 + a_2(s+3)(s+1) + s^2(s+1)b + s^2(s+3)c \Big]$$

$$= \frac{1}{s^2(s+1)(s+3)}\Big[(s^3+4s^2+3s)a_1 + (s^2+4s+3)a_2 + (s^3+s^2)b + (s^3+3s^2)c \Big]$$

$$= \frac{1}{s^2(s+1)(s+3)}\Big[(a_1+b+c)s^3 + (4a_1+a_2+b+3c)s^2 + (3a_1+4a_2)s + 3a_2 \Big]$$

$$= \frac{-2s^3-8s^2+8s+6}{(s+3)(s+1)s^2}$$

Then
$$\begin{cases} 3a_2 = 6 \\ 3a_1+4a_2 = 8 \\ 4a_1+a_2+b+3c = -8 \\ a_1+b+c = -2 \end{cases} \Longrightarrow \begin{cases} a_2 = 2 \\ a_1 = 0 \\ b = 2 \\ c = -4. \end{cases}$$

The final answer is $\boxed{a_1 = 0,\ a_2 = 2,\ b = 2,\ c = -4}$

8. $$\frac{s^3+3s^2+3s+1}{(s^2+2s+5)^2} = \frac{(s+1)^3}{((s+1)^2+4)^2}$$

$$= \frac{a_1(s+1)+b_1\cdot 2}{(s+1)^2+4} + \frac{a_2(s+1)+b_2\cdot 2}{[(s+1)^2+4]^2}$$

$$= \frac{(a_1(s+1)+b_1\cdot 2)((s+1)^2+4) + a_2(s+1)+b_2\cdot 2}{((s+1)^2+4)^2}$$

$$= \frac{a_1(s+1)^3 + b_1\cdot 2\cdot(s+1)^2 + (4a_1+a_2)(s+1) + (8b_1+2b_2)}{((s+1)^2+4)^2}.$$

$$\Rightarrow \quad a_1 = 1, \quad 2b_1 = 0, \quad 4a_1 + a_2 = 0, \quad 8b_1 + 2b_2 = 0.$$

$$\Rightarrow \boxed{a_1 = 1, \quad a_2 = -4, \quad b_1 = 0, \quad b_2 = 0}$$

9. $\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at \quad (a>0).$

Then $\mathcal{L}^{-1}\left(\frac{3}{s^2+4}\right) = \mathcal{L}^{-1}\left(\frac{3}{2}\cdot\frac{2}{s^2+4}\right)$

$= \frac{3}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) = \frac{3}{2}\sin(2t).$

10. $\mathcal{L}^{-1}\left(\frac{n!}{(s-a)^{n+1}}\right) = t^n e^{at} \quad s>a.$

Then $\mathcal{L}^{-1}\left(\frac{4}{(s-1)^3}\right) = \mathcal{L}^{-1}\left(2\cdot\frac{2!}{(s-1)^{2+1}}\right)$

$= 2\mathcal{L}^{-1}\left(\frac{2!}{(s-1)^{2+1}}\right) = 2t^2e^t.$