

MATH 53 HOMEWORK 7 SOLUTION

Section 4.3 problems 27, 30, 37, 40, 42.

Section 4.4 problem 7, 11.

Section 4.5 problems 1, 3, 13, 17.

Problem 4.3.27 The characteristic equation is

$$\lambda^2 + \lambda - 2 = 0 \Rightarrow (\lambda + 2)(\lambda - 1) = 0 \Rightarrow \lambda = 1, -2$$

So the general solution is given by

$$y(t) = C_1 e^t + C_2 e^{-2t} \Rightarrow y'(t) = C_1 e^t - 2C_2 e^{-2t}$$

Using the initial condition, we get

$$\begin{aligned} 1 = y(0) &= C_1 + C_2 \\ 1 = y'(0) &= C_1 - 2C_2 \end{aligned} \Rightarrow C_1 = 1, C_2 = 0$$

so the solution to IVP is

$$y(t) = e^t$$

The solution goes to $+\infty$ as $t \rightarrow \infty$.

Problem 4.3.30 The characteristic equation is

$$\lambda^2 + 4\lambda + 3 = 0 \Rightarrow (\lambda + 3)(\lambda + 1) = 0 \Rightarrow \lambda = -3, -1$$

So the general solution is given by

$$y(t) = C_1 e^{-t} + C_2 e^{-3t} \Rightarrow y'(t) = -C_1 e^{-t} - 3C_2 e^{-3t}$$

Using the initial condition, we get

$$\begin{aligned} 2 = y(0) &= C_1 + C_2 \\ -1 = y'(0) &= -C_1 - 3C_2 \end{aligned} \Rightarrow C_2 = -1/2 \Rightarrow C_1 = 5/2$$

so the solution to IVP is

$$y(t) = \frac{5}{2} e^{-t} - \frac{1}{2} e^{-3t}$$

The solution goes to 0 as $t \rightarrow \infty$.

Problem 4.3.37 The characteristic equation is

$$\lambda^2 + 4\lambda + 4 = 0 \Rightarrow (\lambda + 2)^2 = 0 \Rightarrow \lambda = -2$$

So the general solution is given by

$$y(t) = C_1 e^{-2t} + C_2 t e^{-2t} \Rightarrow y'(t) = -2C_1 e^{-2t} + C_2 e^{-2t} - 2C_2 t e^{-2t}$$

Using the initial condition, we get

$$\begin{aligned} 2 = y(-1) &= C_1 e^2 - C_2 e^2 \\ 1 = y'(-1) &= -2C_1 e^2 + C_2 e^2 + 2C_2 e^2 \end{aligned} \Rightarrow C_2 = 5e^{-2} \Rightarrow C_1 = 7e^{-2}$$

so the solution to IVP is

$$y(t) = 7e^{-2(t+1)} + 5te^{-2(t+1)}$$

The solution goes to 0 as $t \rightarrow \infty$.

Problem 4.3.40 The characteristic equation is

$$2\lambda^2 + \lambda - 4 = 0 \quad \Rightarrow \quad \lambda = \frac{-1 \pm \sqrt{1 + 32}}{4} = \frac{-1 \pm \sqrt{33}}{4}$$

So the general solution is given by

$$y(t) = C_1 e^{\frac{-1+\sqrt{33}}{4}t} + C_2 e^{\frac{-1-\sqrt{33}}{4}t} \quad \Rightarrow \quad y'(t) = \left(\frac{-1+\sqrt{33}}{4}\right)C_1 e^{\frac{-1+\sqrt{33}}{4}t} + \left(\frac{-1-\sqrt{33}}{4}\right)C_2 e^{\frac{-1-\sqrt{33}}{4}t}$$

Using the initial condition, we get

$$\begin{aligned} 0 = y(0) &= C_1 + C_2 \\ 1 = y'(0) &= \left(\frac{-1+\sqrt{33}}{4}\right)C_1 + \left(\frac{-1-\sqrt{33}}{4}\right)C_2 \end{aligned} \quad \Rightarrow \quad C_1 = \frac{2}{\sqrt{33}} \quad \Rightarrow \quad C_2 = -\frac{2}{\sqrt{33}}$$

so the solution to IVP is

$$y(t) = \frac{2}{\sqrt{33}} e^{\frac{-1+\sqrt{33}}{4}t} - \frac{2}{\sqrt{33}} e^{\frac{-1-\sqrt{33}}{4}t}$$

Because $\frac{-1+\sqrt{33}}{4} > 0$, the solution goes to $+\infty$ as $t \rightarrow \infty$.

Problem 4.3.42 The characteristic equation is

$$\lambda^2 + 2\lambda + 2 = 0 \quad \Rightarrow \quad (\lambda + 1)^2 = -1 \quad \Rightarrow \quad \lambda = -1 \pm i$$

So the general solution is given by

$$y(t) = C_1 e^{-t} \cos t + C_2 e^{-t} \sin t \quad \Rightarrow \quad y'(t) = -C_1 e^{-t} \cos t - C_1 e^{-t} \sin t - C_2 e^{-t} \sin t + C_2 e^{-t} \cos t$$

Using the initial condition, we get

$$\begin{aligned} 2 = y(\pi/4) &= e^{-\pi/4} \frac{\sqrt{2}}{2} (C_1 + C_2) \\ -2 = y'(\pi/4) &= e^{-\pi/4} \frac{\sqrt{2}}{2} (-C_1 - C_1) \end{aligned} \quad \Rightarrow \quad C_1 = C_2 = \sqrt{2} e^{\pi/4}$$

so the solution to IVP is

$$y(t) = \sqrt{2} e^{-(t-\pi/4)} \cos t + \sqrt{2} e^{-(t-\pi/4)} \sin t$$

The solution oscillates with decaying amplitude as $t \rightarrow \infty$.

Problem 4.4.7 The mass is given by

$$m = \frac{w}{g} = \frac{3 \text{ lb}}{32 \text{ ft/s}^2}$$

The spring constant k is found from the statement that the mass stretches the spring by 3 in = 1/4 ft

$$k = \frac{3}{1/4} = 12 \text{ lb/ft}$$

Consequently, y satisfies

$$\frac{3}{32} y'' + 12y = 0 \quad \Rightarrow \quad y'' + 128y = 0$$

so we get

$$\lambda = \pm \sqrt{128} = \pm 8\sqrt{2}$$

Thus,

$$y(t) = A \cos(8\sqrt{2}t) + B \sin(8\sqrt{2}t)$$

so

$$y'(t) = 8\sqrt{2}(-A \sin(8\sqrt{2}t) + B \cos(8\sqrt{2}t))$$

Plugging in $t = 0$ and using the given initial condition $y(0) = 1 \text{ in} = 1/12 \text{ ft}$ and $y'(0) = 2 \text{ ft/s}$.

$$\begin{aligned} 1/12 = y(0) &= A \\ 2 = y'(0) &= 8\sqrt{2}B \end{aligned} \quad \Rightarrow \quad A = 1/12, B = \sqrt{2}/8$$

Thus,

$$y(t) = (\sqrt{2}/8) \sin(8\sqrt{2}t) - \frac{1}{12} \cos(8\sqrt{2}t)$$

so the frequency $\omega = 8\sqrt{2}\text{rad/s}$, the period $T = 2\pi/\omega = \sqrt{2}\pi/8$ s, the amplitude $R = \sqrt{2/64 + 1/144} \approx 0.1954$ ft, the phase $\delta = \pi - \arctan((3/\sqrt{2})) \approx 2.0113$.

Problem 4.4.11 (a) The spring constant k can be found by

$$F(\Delta y) = -k\Delta y \Rightarrow -3 = -k0.1 \Rightarrow k = 30$$

The damp coefficient is given by

$$\gamma = F_d/y' = \frac{3 \text{ N}}{5 \text{ m/s}} = 0.6$$

So the equation for $y(t)$ is

$$2y''(t) + 0.6y'(t) + 30y(t) = 0$$

The characteristic equation is

$$2\lambda^2 + 0.6\lambda + 30 = 0 \Rightarrow \lambda = \frac{-0.6 \pm \sqrt{0.6^2 - 4(2)(30)}}{4} = -0.15 \pm 3.87i$$

so

$$y(t) = e^{-0.15t}(A \cos 3.87t + B \sin 3.87t)$$

Thus,

$$y'(t) = -0.15e^{-0.15t}(A \cos 3.87t + B \sin 3.87t) + 3.87e^{-0.15t}(B \cos 3.87t - A \sin 3.87t)$$

Using $y(0) = 0.05$, $y'(0) = 0.1$ m/s, we get

$$A = 0.05, B = (0.1 + 0.15 * 0.05)/3.87 = 0.0277$$

Thus,

$$y(t) = e^{-0.15t}(0.05 \cos 3.87t + 0.0277 \sin 3.87t) \approx 0.057e^{-0.15t} \cos(3.87t - 0.51)$$

From the above expression, one get

$$\nu = 3.87, \quad \nu/\omega_0 = 3.87/\sqrt{30/2} = 0.99925$$

(b) See the back of the text book.

Problem 4.5.1 The characteristic equation is given by

$$\lambda^2 - 2\lambda - 3\lambda = 0 \Rightarrow \lambda = 3, -1$$

so the general solution for the homogeneous equation

$$y_h(t) = C_1e^{3t} + C_2e^{-t}$$

Suppose $y_p(t) = Ae^{2t}$, we get

$$y_p''(t) - 2y_p'(t) - 3y_p(t) = (4A - 4A - 3A)e^{2t} = 3e^{2t} \Rightarrow A = -1$$

so the general solution is

$$y(t) = y_p(t) + y_h(t) = -e^{2t} + C_1e^{3t} + C_2e^{-t}$$

Problem 4.5.3 the characteristic equation is given by

$$\lambda^2 - 2\lambda - 3\lambda = 0 \Rightarrow \lambda = 3, -1$$

so the general solution for the homogeneous equation

$$y_h(t) = C_1e^{3t} + C_2e^{-t}$$

Suppose $y_p(t) = t(At + B)e^{-t} = (At^2 + Bt)e^{-t}$ [we need to multiply by t because Be^{-t} is a solution to the homogeneous equation], so we get

$$y_p'(t) = -(At^2 + Bt)e^{-t} + (2At + B)e^{-t} = (-At^2 - Bt + 2At + B)e^{-t}$$

$$y_p''(t) = (At^2 + Bt - 2At - B - 2At - B + 2A)e^{-t} = (At^2 + (B - 4A)t + 2A - 2B)e^{-t}$$

Thus,

$$-3te^{-t} = y_p''(t) - 2y_p'(t) - 3y_p(t) = ((B - 4A - 2(2A - B) - 3B)t + 2A - 2B - 2B)e^{-t}$$

$$= (-8At + 2A - 4B)e^{-t} \Rightarrow \begin{array}{l} -3 = -8A \\ 0 = 2A - 4B \end{array} \Rightarrow A = 3/8, B = 3/16$$

so the general solution is

$$y(t) = y_p(t) + y_h(t) = \frac{3}{8}t^2e^{-t} + \frac{3}{16}te^{-t} + C_1e^{3t} + C_2e^{-t}$$

Problem 4.5.13 The characteristic equation is given by

$$\lambda^2 + \lambda - 2\lambda = 0 \Rightarrow \lambda = -2, 1$$

so the general solution for the homogeneous equation

$$y_h(t) = C_1e^t + C_2e^{-2t}$$

Suppose $y_p(t) = At + B$, then

$$2t = y_p''(t) + y_p'(t) - 2y_p(t) = A - 2(At + B) = -2At + A - 2B \Rightarrow A = -1, B = -1/2$$

Thus, the general solution is

$$y(t) = y_p(t) + y_h(t) = -t - 1/2 + C_1e^t + C_2e^{-2t}$$

so

$$y'(t) = -1 + C_1e^t - 2C_2e^{-2t}$$

Use the initial conditions, we get

$$\begin{array}{l} 0 = y(0) = -1/2 + C_1 + C_2 = 0 \\ 1 = y'(0) = -1 + C_1 - 2C_2 \end{array} \Rightarrow C_1 = 1, C_2 = -1/2$$

so

$$y(t) = -t - 1/2 + e^t - \frac{1}{2}e^{-2t}$$

Problem 4.5.17 The characteristic equation is given by

$$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$$

so the general solution for the homogeneous equation

$$y_h(t) = C_1 \cos 2t + C_2 \sin 2t$$

Suppose $y_p(t) = t(A \cos 2t + B \sin 2t)$ [we need to multiply by t because $A \cos 2t + B \sin 2t$ is a solution to the homogeneous equation], then

$$\begin{array}{l} y_p'(t) = A \cos 2t + B \sin 2t + 2t(B \cos 2t - A \sin 2t) \\ y_p''(t) = 2(B \cos 2t - A \sin 2t) + 2(B \cos 2t - A \sin 2t) - 4t(A \cos 2t + B \sin 2t) \end{array}$$

so

$$3 \sin 2t = y_p''(t) + 4y_p = 4(B \cos 2t - A \sin 2t) \Rightarrow B = 0, A = -3/4$$

Thus,

$$\begin{array}{l} y(t) = -\frac{3}{4}t \cos 2t + C_1 \cos 2t + C_2 \sin 2t \\ y'(t) = -\frac{3}{4}(\cos 2t - 2t \sin 2t) - 2C_1 \sin 2t + 2C_2 \cos 2t \end{array}$$

Plugging in the initial conditions,

$$\begin{array}{l} 2 = y(0) = C_1 \\ -1 = y'(0) = -\frac{3}{4} + 2C_2 \end{array} \Rightarrow C_1 = 2, C_2 = -\frac{1}{8}$$

Therefore,

$$y(t) = -\frac{3}{4}t \cos 2t + 2 \cos 2t - \frac{1}{8} \sin 2t$$