Solutions to Math 53 Math 53 Practice Final

1. (20 points) Consider the initial value problem $y''(t) - 4y(t) = te^{-t}$ with $y'(0) = -1$ and $y(0) = 0$.
   (a) (8 points) Find the Laplace transform of the solution of this IVP.
   (b) (8 points) Find the solution of this IVP by performing the inverse Laplace transform to the result of the first part.
   (c) (4 points) Find the general solution to $y''(t) - 4y(t) = te^{-t}$.

(a) We have $L\{y''\} = s^2 L\{y\} - sy(0) - y'(0) = s^2 Y(s) + 1$
Hence $L\{y''(t) - 4y(t)\} = (s^2 - 4)Y(s) + 1$. At the same time $L\{te^{-t}\}(s) = L\{t\}(s+1) = \frac{1}{(s+1)^2}$.
We get $Y(s) = \frac{\frac{1}{(s+1)^2} - 1}{(s-2)(s+2)}$.

(b) $Y(s) = \frac{\frac{1}{(s+1)^2} - 1}{(s-2)(s+2)} = \frac{s(s+2)}{(s-2)(s+2)(s+1)^2} = \frac{-s}{(s-2)(s+1)^2}$. We need to do a partial fractions expansion:
$Y(s) = \frac{-s}{(s-2)(s+1)^2} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$ or $-s = A(s+1)^2 + B(s-2)(s+1) + C(s-2)$, so $s = -1$ gives $C = -\frac{1}{3}$, $s = 2$ fives $A = \frac{-2}{9}$ and $s = 0$ gives $B = \frac{2}{9}$. So $y(t) = \frac{-2}{9}e^{2t} + \frac{2}{9}e^{-t} + \frac{-1}{3}te^{-t}$.

(c) General solution to the homogeneous equation is $c_1 e^{2t} + c_2 e^{-2t}$ so the general solution to the inhomogeneous one is $c_1 e^{2t} + c_2 e^{-2t} + \frac{-2}{9}e^{2t} + \frac{+2}{9}e^{-t} + \frac{-1}{3}te^{-t} = k_1 e^{2t} + k_2 e^{-2t} + \frac{2}{9}e^{-t} - \frac{1}{3}te^{-t}$.

2. (20 points) Consider the second order ODE $y'' - 3y' + 2y = 0$.

   (a) (8 points) Explain how to rewrite this differential equation as a 2×2 system. Use this to find the general solution to the homogeneous equation.

   (b) (8 points) Let a be a real number, with $a \neq 1, a \neq 2$. Find (by any method) the solution $y_a(t)$ to the inhomogeneous equation $y'' - 3y' + 2y = e^{at}$ with initial values $y(0) = 0, y(1) = 0$.

   (c) (4 points) Let $y_a(t)$ be the solution from part (b). Compute the limit of $y_a(t)$ as $a \to 1$ and show that it satisfies $y'' - 3y' + 2y = e^t$.

(a) Set $u = y'$ and $v = y$. Then we have $v' = u$ and $u' = 3y' - 2y = 3u - 2v$, so the system is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

The eigenvalues are roots of $\lambda^2 - 3\lambda + 2 = 0$, so $(\lambda - 2)(\lambda - 1) = 0$; this corresponds to solution $c_1 e^{2t} + c_2 e^t$.

(b) We can use either variation of parameters or the method of undetermined coefficients. We test $y = Ce^{at}$ as a particular solution; we must have $C(a^2 - 3a + 2) = 1$, so that $C = \frac{1}{a^2-3a+2}$. The general solution is therefore $y(t) = c_1 e^{2t} + c_2 e^t + Ce^{at}$; we now impose the initial conditions, which give:

$$c_1 + c_2 + C = 0, 2c_1 + c_2 + aC = 0,$$

so that $c_1 = (1 - a)C$ and $c_2 = (a - 2)C$, and the final solution is

$$\frac{(1 - a)e^{2t} + (a - 2)e^t + e^{at}}{a^2 - 3a + 2}.$$

(c) We take a limit via l'Hopital's rule; this gives

$$\lim_{a\to 1} y_a(t) = \lim_{a\to 1} \frac{-e^{2t} + e^t + te^{at}}{2a - 3}|_{a=1} = (-1 - t)e^t + e^{2t}$$

Call this $y(t)$; then $y' = 2e^{2t} - (t + 2)e^t$ and $y'' = 4e^{2t} - (t + 3)e^t$, and we see

$$y'' - 3y' + 2y = e^t((t + 3) - 3(t + 2) + 2(t + 1)) + (4 - 6 + 2)e^{2t} = e^t.$$

3. (20 points) Consider the nonlinear system of differential equations

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(y-4) \\ (x-2)(y-1) \end{pmatrix}$$

(a) (4 points) What are the equillibria (critical points) of this system?

(b) (4 points) Compute the linearizations of this system at all of its equillibria and classify them according to type (node, center, sink, source etc. etc.)

(c) (6 points) Compute the eigenvectors of the linearized systems.

(d) (6 points) Sketch the phase portrait of this non-linear system, indicating the direction of flow and as many other features as you can.

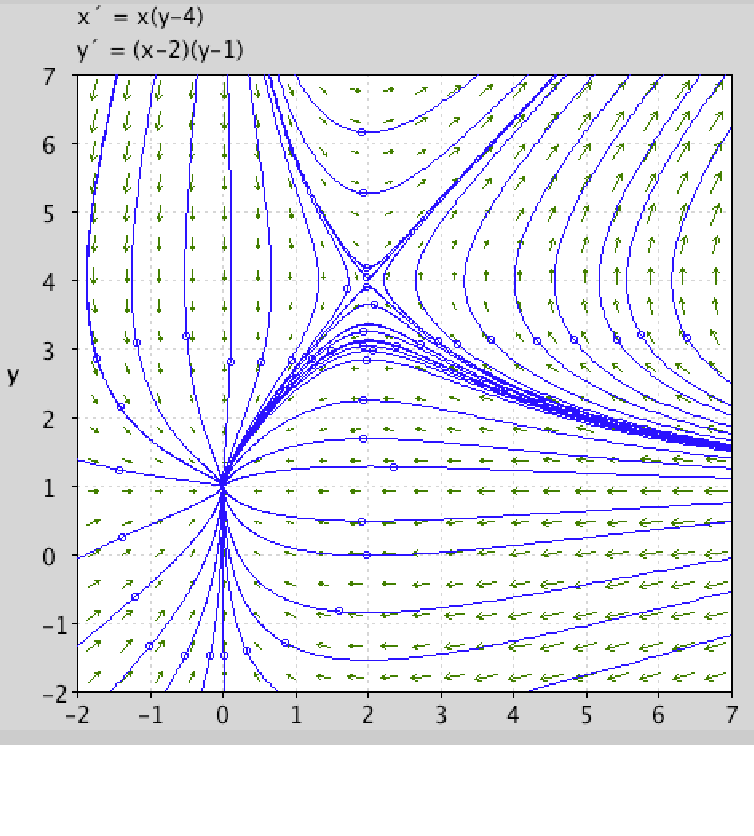
(a) To find critical points we need to solve $x(y-4) = 0$ and $(x-2)(y-1) = 0$. This gives $p = (0,1)$ and $q = (2,4)$.

(b) See below.

(c) We compute the Jacobian matrix $\begin{pmatrix} y-4 & x \\ y-1 & x-2 \end{pmatrix}$.

At p this is $\begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}$. This is a nodal sink with eigenvalue -3 with eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and eigenvalue -2 with eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

At q this is $\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$. This has $\lambda = \pm\sqrt{6}$ The eigenvector for $\sqrt{6}$ is $\begin{pmatrix} 2 \\ \sqrt{6} \end{pmatrix}$ and for $-\sqrt{6}$ its $\begin{pmatrix} -2 \\ \sqrt{6} \end{pmatrix}$. This is a saddle.

(d) 

4. (20 points) (a) (6 points) Define the matrix exponential e^A and compute it for $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(b) (8 points) Consider the system:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$$

where you are not given A, but are given that $e^A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$.

If $x(0) = 3, y(0) = 0$, find $(x(1), y(1))$ and $(x(2), y(2))$.

(c) (6 points, unrelated to (a) and (b)) Consider the one-variable differential equation $x' = 2xt - t^3$. Suppose that $x(1) = 1$. Estimate $x(1.1)$, and explain in words:

(i) whether you expect that your estimate is *larger* or *smaller* than the true solution, and

(ii) how you would produce a more accurate estimate. (You do not need to do the calculation- just describe it.)

(a) The matrix exponential is defined by $e^A = \sum_{i=0}^{\infty} A^i/i!$. For the given matrix, the eigenvectors are $v_1 = (1,1)$ and $v_2 = (1,-1)$, with $Av_1 = v_1$ and $Av_2 = -v_2$. Therefore, $e^A v_1 = ev_1$ and $e^A v_2 = e^{-1}v_2$.

From that, we see that

$$e^A = \frac{1}{2}\begin{pmatrix} e + 1/e & e - 1/e \\ e - 1/e & e + 1/e \end{pmatrix}.$$

(b) The general solution to $\mathbf{v}' = A\mathbf{v}$ with initial condition $\mathbf{v}(0) = \mathbf{v}_0$ is given by $e^{At}\mathbf{v}_0$. So

$$\begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = e^A\begin{pmatrix} x(1) \\ y(1) \end{pmatrix} \text{ and } \begin{pmatrix} x(2) \\ y(2) \end{pmatrix} = e^{2A}\begin{pmatrix} x(1) \\ y(1) \end{pmatrix}$$

Note that $e^{2A} = e^A \cdot e^A$. So $(x(1), y(1)) = e^A \cdot (3,0) = (3,12)$ and $(x(2), y(2)) = e^A \cdot e^A \cdot (3,0) = (39,36)$

(c) Euler's method : $x'(1) = 1$, so $x(1.1) \approx 1.1$. In order to make a better approximation, we'd use Euler's method with a smaller step size.

In order to see if this is an overestimate or underestimate, we compute

$$\frac{d}{dt}x' = \frac{d}{dt}(2xt - t^3) = 2x't + 2x - 3t^2.$$

So, at $t = 1$, $\frac{d}{dt}x' = 2 + 2 - 3 = 1$, so that x' is actually increasing. Thus the truth will be *larger than* 1.1.

5. (20 points) Consider the ODE

$$y' = \frac{1}{2x^2y - 8y}$$

with initial value $y(0) = -\frac{\sqrt{\ln 2}}{2}$.

(a) (10 points) Find the solution of this IVP.

(b) (8 points) What is the maximal interval $a < x < b$ on which this solution exists?

(c) (2 points) Describe the limiting behavior of y as x approaches a and as x approaches b.

(a) This equation is separable with $\frac{dy}{dx} = \frac{1}{2y(x^2-4)}$ so $2ydy = \frac{dx}{x^2-4} = \frac{1}{4}(\frac{1}{x-2} - \frac{1}{x+2})dx$ (you can use partial fractions to get that last equality). Integrating we get $y^2 = \frac{1}{4}(\ln|x-2| - \ln|x+2|) + C$. Plugging in $x = 0$ and $y = -\sqrt{\ln 2}$ we get $\frac{\ln 2}{4} = C$. So $y^2 = \frac{1}{4}(\ln|x-2| - \ln|x+2|) + \frac{\ln 2}{4}$, $2y = \pm\sqrt{\ln\frac{2|x-2|}{|x+2|}}$, $y = \pm\frac{1}{2}\sqrt{\ln\frac{2|x-2|}{|x+2|}}$. The fact that $y(0) < 0$ makes the solution $y = -\frac{1}{2}\sqrt{\ln\frac{2|x-2|}{|x+2|}}$.

(b) The solution exists on the interval $-2 < x < 2/3$.

One way to see this is by direct analysis. Near $x = 0$ we have $|x-2| = 2-x$ and $|x+2| = 2+x$, so the solution can be written as $y = -\frac{1}{2}\sqrt{\ln\frac{2(2-x)}{x+2}}$. The only things that could go wrong:

- $x + 2$ could be zero, so we're dividing by zero; this happens when $x = -2$.
- $\frac{2(2-x)}{x+2}$ could be negative, so we would be taking the logarithm of a negative number. This happens only when $x > 2$.
- $\ln\frac{2(2-x)}{2+x}$ could be negative, so we are taking the square root of a negative number. That happens when $\frac{2(2-x)}{2+x} < 1$, i.e. $4 - 2x < 2 + x$ or $x > 2/3$.

So together this shows that the solution is defined in $-2 < x < 2/3$.

Another way to get to the same result is by using Theorem 2.3.2 in the book. It says that, if $a < x < b$ is the largest interval where the solution exists, one of the following must happen at $x = a$ and $x = b$: there is a discontinuity in $\frac{1}{2x^2y-8y}$, or its partial $\partial f/\partial y$, or y becomes infinite. That happens if $x = 2$ or $x = -2$ or $y = -\frac{1}{2}\sqrt{\ln\frac{2|x-2|}{|x+2|}} = 0$ or $y = -\frac{1}{2}\sqrt{\ln\frac{2|x-2|}{|x+2|}} = \pm\infty$. The last two conditions give $2|x-2| = |x+2|$ and $|x+2| = 0$ or $|x-2| = 0$. Since we start with $-2 < x < 2$ we must stay there, and $|x+2| = x+2$ and $|x-2| = 2-x$, so the other condition to check is $(x+2) = 2(2-x)$ or $3x = 2$, $x = \frac{2}{3}$. So $-2 < x < \frac{2}{3}$.

(c) When x approaches -2 we get $y = -\frac{1}{2}\sqrt{\ln\frac{2|x-2|}{|x+2|}}$ approaches $-\ln +\infty = -\infty$ and as x approaches $\frac{2}{3}$ we get y approaches 0.

6. (20 points) Consider the first order equation

$$\frac{dy}{dt} = p(t)y + q(t)$$

   (a) (4 points) Describe a situation which is (at least roughly) modelled by this equation. (For example, y might be population; interpret what $p(t)$ and $q(t)$ represent.)

   (b) (3 points) Suppose that $y(0) > 0$ and $q(t) > 0$, but you are not told anything about $q(t)$. (**Correction:** this should have been $p(t)$.) Must $y(100)$ be greater than zero?

   (c) (8 points) Suppose that $p(t) = 1/t$ and $q(t) = t^2e^{-t^2}$. Find the solution with initial condition $y(1) = 0$.

   (d) (5 points) Now suppose we replace $q(t)$ with the function

$$q(t) = \begin{cases} 0, t \leq 2. \\ t^2e^{-t^2}, t > 2 \end{cases}$$

   Find $y(3)$.

(a) In the population example, $p(t)$ represents the natural growth rate (due to births and deaths) and $q(t)$ represents the external influx/outflux (due to immigration and emigration, say).

(b) Yes. We can see this from the explicit form of the solution with integrating factors, or by thinking about the fact that this models a population with only immigration and no emigration (because $q(t) > 0$); whether or not $p(t) > 0$, the population will never become negative!

(c) We use the integrating factor $e^{-\int t^{-1}} = 1/t$. The equation becomes

$$(y/t)' = te^{-t^2},$$

Now $\int te^{-t^2} = -\frac{1}{2}e^{-t^2} + c$, so we get

$$y = \frac{-t}{2}e^{-t^2} + ct.$$

The initial condition turns into $c = e^{-1}/2$, so $y = \frac{t}{2}(e^{-1} - e^{-t^2})$.

(d) (**Corrected**) One way to solve this is to first solve the equation up to time 2. The solution for $t \leq 2$ satisfies $(y/t)' = 0$, so that $y = Ct$; because of the initial condition $y(1) = 0$, we must have $C = 0$, so that $y(2) = 0$.

We can now use the same solution from part (b), but with initial condition $y(2) = 0$: this gives

$$-e^{-4} + 2c = 6 \implies c = \frac{1}{2}e^{-4}$$

and so our solution at time $t = 3$ is given by

$$y(3) = (-3/2)e^{-9} + (3/2)e^{-4}.$$

7. (20 points) (a) (3 points) Consider the system described by the second order ODE $y'' + 5y = 0$. Find the general solution of this ODE. What is the natural frequency of this system?

   (b) (7 points) A driving force of $3\sin(5t)$ is applied to this system. Find a particular solution and the general solution of the resulting ODE.

   (c) (10 points) Damping is added to the system, resulting in $y''+2y'+5y = 3\sin(5t)$. Find a particular solution of this ODE. What is the amplitude of oscillations for this system for large t?

(a) The general solution is given by $a\cos(\sqrt{5}t) + b\sin(\sqrt{5}t)$ (or equivalently $A\cos(\sqrt{5}t - \phi)$). The natural is $\frac{\sqrt{5}}{2\pi}$.

(Note: both MM and AV sometimes said "the frequency is $\sqrt{5}$" in this situation in lectures; although $\sqrt{5}/(2\pi)$ is strictly correct, we would accept $\sqrt{5}$ too.)

frequency is $\sqrt{5}$.

(b) The system is $y'' + 5y = 3\sin(5t)$. We guess $y(t) = C\sin(5t)$ to get $C(-25 + 5) = 3\sin(5t)$ so $C = -\frac{3}{20}$ and $y_p(t) = -\frac{3}{20}\sin(5t)$ and $y(t) = -\frac{3}{20}\sin(5t) + a\cos(\sqrt{5}t) + b\sin(\sqrt{5}t)$.

Note: in general you should guess $C\sin(5t) + D\cos(5t)$; in this case if you do this you will end up getting $D = 0$.

(c) We give several ways to do this.

Solution 1) We write the driving force as $Re(3e^{5ti})$, solve the complex version and take the real part. For $y'' + 2y' + 5y = 3e^{5ti}$ we guess $y = Ce^{5ti}$ and get $(-25 + 10i + 5)Ce^{5ti} = 3e^{5ti}$, so $C = \frac{3}{-20+10i}$ which gives $\frac{3}{-20+10i}e^{5ti}$. The complex solution goes in a circle of radius $|\frac{3}{-20+10i}|$ so the real part oscillates with the same amplitude $|\frac{3}{-20+10i}|$ which is

$$\frac{3}{|-20+10i|} = \frac{3}{10\sqrt{5}} = \frac{3}{50}\sqrt{5}.$$

The solution itself is $Re(\frac{3}{-20+10i}e^{5ti}) = \frac{3(-20\cos(5t)-10\sin(5t))}{500} = \frac{-3\cos(5t)-6\sin(5t)}{50}$.

Solution 2) (more work) We guess $A\cos(5t) + B\sin(5t)$ to get $-25A\cos(\omega t) - 25B\sin(\omega t) + 10B\cos(\omega t) - 10\sin(\omega t) + 5A\cos(\omega t) + 5B\sin(\omega t) = \sin(5t)$, to get $-25A + 5A + 10B = 0$ and $-25B + 5B - 10A = 3$ so $B = 2A$ and $-50A = 3$ or $A = \frac{-3}{50}$ and $B = \frac{-6}{50}$. The solution is $\frac{-3}{50}\cos(5t) + \frac{-6}{50}\sin(5t)$. Its amplitude is $\sqrt{(\frac{-3}{50})^2 + \frac{-6}{50})^2} = \frac{3}{50}\sqrt{5}$.

8. (20 points)

 (a) (10 points) Consider the system of differential equations given by $x' = ax + 3y, y' = 2x + y$. Classify the type of critical point for each value of a. For $a = 2$ draw a phase portrait (noting eigenvector directions and trajectory directions).

 (b) Suppose that the 2×2 system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$$

 has $x(t) = e^t \sin(t) + e^t \cos(t), y(t) = e^t \cos(t)$ as a solution.

(i, 5 points) Determine the eigenvalues of A.

(ii, 5 points) Determine A itself.

(a) The matrix is $A = \begin{pmatrix} a & 3 \\ 2 & 1 \end{pmatrix}$ with characteristic polynomial

$$(a - \lambda)(1 - \lambda) - 6 = 0 \implies \lambda^2 - (a + 1)\lambda + (a - 6) = 0.$$

The discriminant is $(a + 1)^2 - 4(a - 6) = a^2 - 2a + 25 = (a - 1)^2 + 24$; it's always positive, so we always have real roots.

The product of the two roots is $a - 6$, so the roots have the same sign for $a > 6$ and opposite signs for $a < 6$.

Finally the sum of the two roots is $(a+1)$, so, when $a > 6$, the roots are positive. In conclusion,

- $a > 6$: nodal source
- $a < 6$: saddle

When $a = 2$ the eigenvalues are the roots of $\lambda^2 - 3\lambda - 4 = 0$, that is $\lambda_1 = 4$ and $\lambda_2 = -1$. The eigenvectors then solve

$$\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} \mathbf{v}_1 = 0,$$

so that $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, and

$$\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \mathbf{v}_2 = 0,$$

so that $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$. (Phase portrait to be added; looks like Figure 3.3.6 from the book.)

(b) We know that, if the eigenvalues of A are $\lambda \pm i\nu$, the solutions look like combinations of $e^{\lambda t}\cos(\nu t)$ and $e^{\lambda t}\sin(\nu t)$. So the eigenvalues here must be $1 \pm i$.

Differentiating x and y, we get

$$\frac{d}{dt}\begin{pmatrix} 2e^t \cos(t) \\ e^t \cos(t) - e^t \sin(t) \end{pmatrix} = A\begin{pmatrix} e^t \cos(t) + e^t \sin(t) \\ e^t \cos(t) \end{pmatrix}$$

Equating coefficients of cos and sin, we get

$$\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} = A\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

so we get (after inverting the right-hand matrix and multiplying) that $A = \begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix}$.