EXERCISES, 1

Do any six questions.

(1) Let k be a field. Find, with proof, all subrepresentations of k^n as an S_n-representation.

(2) Find all the (isomorphism classes of) irreducible representations of the additive group $\mathbf{Z}/23\mathbf{Z}$ over the field $\mathbf{F}_2$. (Hint: the group algebra decomposes as a sum of fields.)

(3) Suppose G is a p-group and K a field of characteristic p.
  (a) Prove that an irreducible representation of G over K is trivial.
  (b) Prove that G embeds in the subgroup of upper triangular matrices in $\mathrm{GL}_n(K)$, for some n.

(4) Let G be a finite group and H a subgroup. Let $A \subset \mathbf{C}G$ be the subalgebra of elements $\{\alpha : h\alpha h' = \alpha\}$ for all h, h'. For any complex representation V of G, we denote by V^H the elements of V that are fixed by H.
  (a) Prove that for $v \in V^H$ and $a \in A$, we have also $a \cdot v \in V^H$.
  (b) Prove that if V is irreducible as a G-representation, then V^H is irreducible (i.e., simple) as an A-module.
  (c) Prove that if V, W are two irreducible G-representations for which V^H and W^H are isomorphic nontrivial A-representations, then V is isomorphic to W.

  Hint for part(c): Let $\theta : V^H \to W^H$ an A-module isomorphism. Let S be the smallest G-invariant subspace of $V \oplus W$ containing the graph of θ. Then $S \neq V \oplus W$. Now the projection maps $S \to V, S \to W$ are both isomorphisms; thus S defines the graph of an isomorphism between V and W.

  Remark. In fact, the association $V \mapsto V^H$ gives a bijection between irreducible representations of G with an H-fixed vector, and simple modules for A; can you describe the inverse?

(5) Let G be a group and $0 \to F_1 \to F_2 \to \cdots \to F_k \to 0$ an exact complex of G-representations over a field k. Let F_j^{ss} be the semisimplification of F_j. Prove that $\bigoplus_{j \text{ odd}} F_j^{ss}$ and $\bigoplus_{j \text{ even}} F_j^{ss}$ are isomorphic.

(6) Let G be a finite group and $\rho : G \to \mathrm{GL}(V)$ a representation of G on the finite-dimensional $\mathbf{Q}$-vector space V. Prove that we can choose a basis for V so that every $\rho(g)$ acts by a matrix with *integral entries*.

  Hint. Choose any basis $e_1, \ldots, e_n$, let $L = \sum \mathbf{Z}e_i$, and "average" L to make it G-invariant.

(7) We continue with the setup of the prior problem. Suppose $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ are two different bases for V so that $\rho(G)$ has integral entries; let $\sigma_1, \sigma_2 : G \to \mathrm{GL}_n(\mathbf{Z})$ be the corresponding homomorphisms. Let $\overline{\sigma_1}, \overline{\sigma_2}$ be the corresponding representations

$$\overline{\sigma_j} : G \xrightarrow{\sigma_j} \mathrm{GL}_n(\mathbf{Z}) \to \mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z}).$$

  Prove that $\overline{\sigma_1}$ and $\overline{\sigma_2}$ – considered as representations of G over the field $\mathbf{Z}/p\mathbf{Z}$ – have isomorphic semisimplifications (in other words, they have the same composition factors with the same multiplicity; they need not, in general, be actually isomorphic).

  Hint. Let L, L' be the $\mathbf{Z}$-modules spanned by $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$; it suffices to check in the case when $L' \subset L$ (why?); now let X be the finite abelian group L/L', on which G acts, so that one has an exact sequence

$$\{x \in X : px = 0\} \to L'/pL' \to L/pL \to X/pX;$$

use (a slight variant of) 5.

(8) Let k and k' be two algebraically closed fields whose characteristic does not divide $|G|$. Let m be a positive integer. Prove that the number of irreducible representations of G of dimension m over k equals the number of irreducible representations of G of dimension m over k'.