MATH 210B. HOMEWORK 8

1. Let α be an algebraic integer, with monic minimal polynomial $f \in \mathbf{Q}[x]$. Thus $\mathbf{Z}[\alpha]$ is an order in the field $K = \mathbf{Q}(\alpha)$.

(i) Prove that

$$\operatorname{disc} \mathbf{Z}[\alpha] = \operatorname{disc}(f),$$

i.e. the two uses of the word "discriminant" are compatible!

(ii) Compute the ring of integers $\mathcal{O}_K$ in $K = \mathbf{Q}(\alpha)$, where α is a root of $\alpha^3 + \alpha + 34 = 0$. (Hint: begin by using (i) and the results of the prior homework to show that the index of $\mathbf{Z}[\alpha]$ in $\mathcal{O}_K$ is at most 4.)

2. Let $f, g \in \mathbf{Q}[x]$ be irreducible. Explain how to algorithmically test *in a finite time* whether $\mathbf{Q}[x]/f$ and $\mathbf{Q}[x]/g$ are isomorphic.

Hint: reduce to the case where f, g are monic. Let $K = \mathbf{Q}[x]/f$. From a isomorphism $\mathbf{Q}[x]/g \simeq K$ we obtain an element $\beta \in \mathcal{O}_K$ with minimal polynomial g. You may assume we can algorithmically compute a $\mathbf{Z}$-basis $\alpha_1, \dots, \alpha_r$ for $\mathcal{O}_K$; explain how to bound the coefficients of β.

3. Prove or disprove: if R is a commutative ring and $R^n \to R^m$ is a surjective polynomial map, then $n \geq m$.

4. For this question, we use the fact that $k[x_1, \dots, x_n]$ is a unique factorization domain; you might want to review the proof of that.

(i) Prove that the *maximal* proper irreducible algebraic sets of K^n are exactly those of the form $f = 0$ where $f \in k[x_1, \dots, x_n]$ is irreducible.

(ii) Prove that such a set has dimension $n - 1$.