

MATH 210B. HOMEWORK 5

1. Compute all the common zeroes of $y^3 + xy + 1$ and $x^3 + y^3 = x^2 + y^2$. You should express your answer so that there is no ambiguity in the y -value corresponding to a given x -value or vice versa (e.g. something like this would do: $(x, y) = (\alpha, \alpha^{-1})$, where α varies over solutions to $\alpha^3 - 3\alpha + 1 = 0$).

2. The polynomials $f_1 = x^2 + y^2 + 1$, $f_2 = xy + 3y^2 + y + 2$ and $f_3 = y^2 - xy - x + 3$ have no common zero in \mathbf{C}^2 ; find $g_1, g_2, g_3 \in \mathbf{C}[x, y]$ such that $f_1 g_1 + f_2 g_2 + f_3 g_3 = 1$. Find all primes where f_1, f_2, f_3 have a common zero in $\overline{\mathbf{F}}_p^2$.

3. In Wednesday's class, I defined the resultant of a degree n polynomial f and a degree m polynomial g to be a certain $(n + m) \times (n + m)$ determinant, namely, the determinant of the map

$$(p_0, \dots, p_{m-1}, q_0, \dots, q_{n-1}) \mapsto \text{coefficients of } \left(\sum p_i x^i \cdot f + \sum q_j x^j \cdot g \right)$$

Prove that, when f, g are monic, then the resultant of f, g coincides up to sign with $\prod_{\alpha: f(\alpha)=0} g(\alpha)$, where the product over roots is taken with multiplicity.

4. (i) Define, as in class, a constructible set to be a finite union of sets of the form $C \cap U$, where C is open and K is closed. Verify that constructible sets are precisely those sets obtained from open and closed sets by taking finite intersections and unions.

(ii) Give an example to show that “image of a constructible set by a polynomial map is constructible” fails if the field is not algebraically closed. (Nonetheless, there is a version over the field \mathbf{R} : look up semialgebraic sets.)

5. If R is a ring, an element $e \in R$ is called *idempotent* if $e^2 = e$.

(i) Using e and $1 - e$, show that specifying an idempotent is ‘the same’ as specifying an ordered decomposition $R \simeq R_1 \times R_2$ for rings R_1, R_2 .

(ii) If $Z \subset k^n$ is an affine algebraic set, with ideal I , interpret (with proof!) idempotents in $k[x_1, \dots, x_n]/I$ in terms of connected components of Z . Deduce that $k[Z]$ has only finitely many idempotents.

6. Suppose that $Z_1, \dots, Z_r \cdots \subset \mathbf{C}^n$ is a possibly infinite sequence of algebraic sets.

(i) Prove (using Noetherian-ness) that $\cap Z_i$ is actually equal to a finite intersection: $\cap_{i \in S} Z_i$, where S is finite.

(ii) Now suppose that each Z_i is given by the zero-set of a polynomial of degree k . Give an explicit upper bound for $|S|$ in terms of n and k .

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7. Prove that the union of all *minimal* prime ideals in a ring R coincides with the set of zero-divisors in R .