

MATH 210B. HOMEWORK 5

1. Let L/K be a separable extension of fields. Let N be a normal closure, and let $\sigma_1, \dots, \sigma_k$ be the set of embeddings of L into N . Show that, for $x \in L$, the trace is given by $\sum_i \sigma_i(x)$ and the norm by $\prod_i \sigma_i(x)$.

2. Let L_1 and L_2 be two finite extensions of k inside an extension field L/k .

(i) Construct a natural surjection of k -algebras $h: L_1 \otimes_k L_2 \rightarrow L_1 L_2$, and deduce that $[L_1 L_2 : k] \leq [L_1 : k][L_2 : k]$ with equality if and only if h is an isomorphism.

(ii) Prove that if $\gcd([L_1 : k], [L_2 : k]) = 1$ then equality holds in (i), and that the converse is false.

(iii) Prove that if $a \in L$ is algebraic over k and $[k(a) : k]$ is relatively prime to $[L_1 : k]$ then $[L_1(a) : L_1] = [k(a) : k]$. Deduce that in such cases the minimal polynomial for a over k is irreducible over L_1 and hence serves as the minimal polynomial for a over L_1 .

3. (Corrected!) Let $E \subset E' \subset \mathbf{R}$, where E'/E can be decomposed as $E \subset E_1 \subset E_2 \subset \dots \subset E_n = E'$, and $E_j = E_{j-1}(\alpha)$, where some power of α belongs to E_{j-1} . Prove that E'/E does not contain a subextension K/E which is Galois of odd prime degree.

4. Over a field $k = \bar{k}$ with $\text{char}(k) \neq 2$, decompose $\underline{Z}(y^4 - x^2, y^4 - x^2 y^2 + x y^2 - x^3) \subset k^2$ and $\underline{Z}(u^2 + v^2 - 1, u^2 - w^2 - 1) \subset k^3$ into irreducible components. Also show that $V := \{(t, t^2, t^3) \in k^3 \mid t \in k\}$ is closed in k^3 and compute $\underline{I}(V)$.

5. Let k be a field that isn't necessarily algebraically closed. Show that every maximal ideal in $k[x_1, \dots, x_n]$ is of the form $\{f : f(a_1, \dots, a_n) = 0\}$, where $a_i \in L$, a finite extension of k .

6. Suppose that $P_1, \dots, P_n \in \mathbf{Q}[x_1, \dots, x_n]$ are such that $\underline{P} : (P_1, \dots, P_n)$ gives an injection $\mathbf{C}^n \rightarrow \mathbf{C}^n$. Show that \underline{P} defines a bijective map $k^n \rightarrow k^n$ whenever k is a finite field of sufficiently large characteristic. (In a future problem set, we'll extend this to the case when $P_i \in \mathbf{C}[x_1, \dots, x_n]$, and also deduce that $\underline{P} : \mathbf{C}^n \rightarrow \mathbf{C}^n$ is actually bijective.)

7. (not easy) Suppose that $f_1, \dots, f_a \in \mathbf{C}[x_1, \dots, x_b]$ have degree $\leq c$ and have no common zero in \mathbf{C}^b . Prove the existence of an integer N , depending on a, b, c such that we may find $g_i \in \mathbf{C}[x_1, \dots, x_b]$ with degree $\leq N$ and $\sum g_i f_i = 1$.