MATH 210B. HOMEWORK 3

1. Let L/K be an extension of finite fields.
   (i) Prove the primitive element theorem in this case.
   (ii) Prove that the norm from L to K is surjective.

2. Let $K \subset L$ be a Galois extension.
   (i) Prove that an intermediate field $E \subset K$ is normal over K if and only if $\mathrm{Gal}(L/E)$ is normal in $\mathrm{Gal}(L/K)$.
   (ii) Give an example to show that normality is not transitive (i.e., $K \subset L$ normal and $L \subset E$ normal doesn't mean that $K \subset E$ is normal.)

3. Let $L \supset K$ be a normal extension, and let $G = \mathrm{Gal}(L/K)$.
   (i) Prove that L is separable over L^G, and L^G is purely inseparable over K.
   (ii) Prove that the maximal separable subextension L' of L is in fact Galois over K.
   (iii) Prove that the natural map $L' \otimes_K L^G \to L$ is an isomorphism.

4.
   (i) For $\mathbf{Q}(\alpha)/\mathbf{Q}$ where $\alpha^3 - 5 = 0$ compute the norm of $c_0 + c_1\alpha + c_2\alpha^2$ ($c_i \in \mathbf{Q}$) .
   (ii) Suppose that $P_1, \ldots, P_r \in \mathbf{Q}[x_1, \ldots, x_n]$ are polynomials with a unique common zero $(z_1, \ldots, z_n)$ in $\mathbf{C}^n$. Prove this common zero lies in $\mathbf{Q}^n$. Give an example to show that this is not true if $\mathbf{Q} \subset \mathbf{C}$ is replaced by an arbitrary $K \subset L$ with L algebraically closed.

5. Suppose L is an algebraically closed field of characteristic 0. Show that L doesn't have an automorphism of order p, if $p > 2$. (Hint: as a first step, show that the fixed field of such an automorphism would contain the pth roots of unity. Next, think about norms.)