MATH 210B. HOMEWORK 3

1. Let $\alpha_1, \dots, \alpha_5$ be the roots of $x^5 - x + 1$. This has Galois group S_5; you can assume this without proof. Write down a polynomial p in $\mathbf{Q}[X]$ of degree 6 so that

$$\mathbf{Q}(\beta) \subset \mathbf{Q}(\alpha_1, \dots, \alpha_5)$$

where β is a root of p. You do not need to write a proof but explain briefly how you found p.

2. Let α be a root of $x^4 - x^3 - 4x^2 + 4x + 1$. Prove that $\mathbf{Q}(\alpha)$ is Galois over $\mathbf{Q}$ and determine its Galois group.

3. Let α be a root of $x^6 - 3x^5 + 5x^4 - 5x^3 + 5x^2 - 3x + 1$. Prove that $\mathbf{Q}(\alpha)$ is Galois over $\mathbf{Q}$ and the Galois group is nonabelian of order 6.

4. For $N \geq 1$, a *primitive Nth root of unity* over a field k is an element ζ in an extension of k so that $\zeta^N = 1$ and the multiplicative group generated by ζ has order exactly N.
(*i*) If N is divisible by the characteristic of k (in particular, k must have positive characteristic), then show that no primitive Nth root of unity exists over k.
(*ii*) If N is not divisible by the characteristic of k (always the case if k has characteristic 0), then prove that a primitive Nth root of unity exists over k. In addition, show that an extension L/k contains a primitive Nth root of unity over k if and only if it contains a splitting field for $X^N - 1 \in k[X]$. In this case, show that the number of primitive Nth roots of unity over k in L is $\varphi(N) = |(\mathbf{Z}/N)^\times|$.

5. Let k be a field containing a primitive Nth root of unity. Let $\mu_N \subseteq k^\times$ be the subgroup of Nth roots of 1 in $k^\times$ (cyclic of order N). For $a \in k^\times$, let F/k be a splitting field of $X^N - a$.

(*i*) Show that $F = k(\alpha)$ where α is any root to $X^N - a$ in F. Also, for any $\sigma \in \mathrm{Gal}(F/k)$, show that $\sigma(\alpha)/\alpha \in \mu_N$ and that this is *independent* of the choice of α.

(*ii*) Fix a choice of α as in (*i*). Show that the map $\mathrm{Gal}(F/k) \to \mu_N$ given by $\sigma \mapsto \sigma(\alpha)/\alpha$ is an injective group homomorphism. In particular, $\mathrm{Gal}(F/k)$ is *abelian*. Also, show that this map is an isomorphism if and only if $X^N - a$ is irreducible in $k[X]$.