MATH 210B. HOMEWORK 2

In this homework, for f a polynomial of degree n, we use "Galois group of f" as a shorthand for the Galois group of the splitting field of f, considered as a subgroup of S_n via its action on the n roots of f.

1. Let $f \in \mathbf{Q}[X]$ be a polynomial with distinct roots $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$, and define the discriminant

$$\Delta_f = \prod_{i<j} (\alpha_i - \alpha_j)^2 \in \mathbf{Q}.$$

Prove that this quantity is a square *if and only if* the Galois group of f is contained in A_n. Deduce that $\mathbf{Q}[x]/f(x)$ is Galois for $f = X^3 - 3X - 1$ but not for $f = X^3 - 3X - 3$.

2. Let $L/\mathbf{Q}$ be a splitting field for $X^5 - 2 \in \mathbf{Q}[X]$. Show that $L = \mathbf{Q}(\alpha, \zeta)$ with $\alpha^5 = 2$ and $\zeta^5 = 1$ with $\zeta \neq 1$, and that $[L : \mathbf{Q}] = 20$. Rigorously describe $\mathrm{Gal}(L/\mathbf{Q})$ as a semi-direct product, and determine all intermediate fields and containments among them.

3. Prove that the Galois group of $X^4 - 10X^2 + 1 \in \mathbf{Q}[X]$ is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and identify all intermediate fields.

4. Prove that $f = 2X^5 - 10X + 5 \in \mathbf{Q}[X]$ has Galois group S_5. (Hint : check that f is irreducible. Explain why this means that the Galois group acts transitively on the 5 roots. Use complex conjugation to show the Galois group contains a transposition. Deduce that $\mathrm{Gal}(L/\mathbf{Q}) = S_5$.)

5. (i) For a commutative ring R and a pair of R-alg ebras A and A', prove that $A \otimes_R A'$ has a unique R-algebra structure with identity $1 \otimes 1'$ such that $(a_1 \otimes a_1')(a_2 \otimes a_2') = (a_1 a_2) \otimes (a_1' a_2')$ and the R-algebra structure recovers the underlying R-module structure.

(ii) Prove that $V \otimes_k V'$ is nonzero for any nonzero vector spaces V and V' over a field k, and deduce that if K and K' are extensions of k then $K \otimes_k K'$ is a nonzero k-algebra. Use a maximal ideal of this algebra to construct an extension field F of k into which both K and K' embed as subfields *over* k.

(iii) Use (ii) to show that any two fields of the same characteristic can be realized as subfields of a common field. (Hint: take $k = \mathbf{Q}$ or $\mathbf{F}_p$.)