In this homework, for \( f \) a polynomial of degree \( n \), we use “Galois group of \( f \)” as a shorthand for the Galois group of the splitting field of \( f \), considered as a subgroup of \( S_n \) via its action on the \( n \) roots of \( f \).

1. Let \( f \in \mathbb{Q}[X] \) be a polynomial with distinct roots \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \), and define the discriminant
\[
\Delta_f = \prod_{i < j} (\alpha_i - \alpha_j)^2 \in \mathbb{Q}.
\]
Prove that this quantity is a square if and only if the Galois group of \( f \) is contained in \( A_n \). Deduce that \( \mathbb{Q}[x]/f(x) \) is Galois for \( f = X^3 - 3X - 1 \) but not for \( f = X^3 - 3X - 3 \).

2. Let \( L/\mathbb{Q} \) be a splitting field for \( X^5 - 2 \in \mathbb{Q}[X] \). Show that \( L = \mathbb{Q}(\alpha, \zeta) \) with \( \alpha^5 = 2 \) and \( \zeta^5 = 1 \) with \( \zeta \neq 1 \), and that \( [L : \mathbb{Q}] = 20 \). Rigorously describe \( \text{Gal}(L/\mathbb{Q}) \) as a semi-direct product, and determine all intermediate fields and containments among them.

3. Prove that the Galois group of \( X^4 - 10X^2 + 1 \in \mathbb{Q}[X] \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) and identify all intermediate fields.

4. Prove that \( f = 2X^5 - 10X + 5 \in \mathbb{Q}[X] \) has Galois group \( S_5 \). (Hint: check that \( f \) is irreducible. Explain why this means that the Galois group acts transitively on the 5 roots. Use complex conjugation to show the Galois group contains a transposition. Deduce that \( \text{Gal}(L/\mathbb{Q}) = S_5 \).)

5. (i) For a commutative ring \( R \) and a pair of \( R \)-algebras \( A \) and \( A' \), prove that \( A \otimes_R A' \) has a unique \( R \)-algebra structure with identity \( 1 \otimes 1' \) such that \((a_1 \otimes a_1')(a_2 \otimes a_2') = (a_1 a_2) \otimes (a_1' a_2') \) and the \( R \)-algebra structure recovers the underlying \( R \)-module structure.

(ii) Prove that \( V \otimes_k V' \) is nonzero for any nonzero vector spaces \( V \) and \( V' \) over a field \( k \), and deduce that if \( K \) and \( K' \) are extensions of \( k \) then \( K \otimes_k K' \) is a nonzero \( k \)-algebra. Use a maximal ideal of this algebra to construct an extension field \( F \) of \( k \) into which both \( K \) and \( K' \) embed as subfields over \( k \).

(iii) Use (ii) to show that any two fields of the same characteristic can be realized as subfields of a common field. (Hint: take \( k = \mathbb{Q} \) or \( \mathbb{F}_p \).)