Hand in problems 2 and 5–8.

In this homework: P will always denote a prime ideal of R, and R_P, M_P etc. refers to localization at P. If M, N are R-modules, let $\mathrm{Hom}(M, N)$ be the group of all module homomorphisms $M \to N$; it has a natural structure of R-module itself, via the rule: for $f \in \mathrm{Hom}(M, N)$, the homomorphism rf sends m to $rf(m)$. We will use (in questions 1 and 2) the notion of projective R-module, which you might want to review. A "number field" is a finite extension of $\mathbf{Q}$.

1. (i) Verify, as stated in class, that localization at a prime ideal is an exact functor.

(ii) Prove that $\mathrm{Hom}(M, N)_P \simeq \mathrm{Hom}_{R_P}(M_P, N_P)$ *if M is finitely presented, i.e., $M \simeq R^n / K$ with K finitely generated.*

(iii) Suppose M is finitely presented. Prove that M_P is projective for all P if and only if M is projective.

2. Let R be a Dedekind ring and let $M \subset R^n$ be a submodule. Prove there are ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ such that $M \simeq \bigoplus \mathfrak{a}_i$ as modules. (Hint: induct using 1(iii).)

3. Let $f : R \to S$ be a homomorphism of rings, and $\mathfrak{p}$ a prime ideal of R. Prove that there exists a prime ideal $\mathfrak{q}$ of S with $f^{-1}\mathfrak{q} = \mathfrak{p}$ if and only if

$$S \otimes_R k(\mathfrak{p})$$

is nonzero. Here $k(\mathfrak{p})$ is the quotient field of $R/\mathfrak{p}$.

(Hint: The prime ideals $\mathfrak{q}$ correspond exactly to prime ideals of S that both contain $\mathfrak{p}$ *and* are disjoint from $T := \{R - \mathfrak{p}\}$, i.e. to prime ideals of $S'/\mathfrak{p}S'$ where $S' = T^{-1}S$. Explain why $S'/\mathfrak{p}S'$ is isomorphic to $S \otimes_R k(\mathfrak{p})$.)

4. Suppose $R \subset S$ are rings and S is *free* as an R-module. Let $\mathfrak{p}_1 \subset \mathfrak{p}_2$ be prime ideals of R, and let $\mathfrak{q}_2$ be a prime ideal of S above $\mathfrak{p}_2$.

(i) Put $M = R/\mathfrak{p}_1$. Prove that $S_{\mathfrak{q}_2} \otimes_R M \neq 0$.

(ii) Let k be the quotient field of M. Prove that $S_{\mathfrak{q}_2} \otimes_R k \neq 0$.

(iii) Prove that there exists a prime ideal $\mathfrak{q}_1 \subset \mathfrak{q}_2$ above $\mathfrak{p}_1$.

(Hint: For (ii), consider the exact sequence $M \to k \to k/M$. Tensor it with S and then localize at $\mathfrak{q}_2$. For (iii), use problem 2.)

5. Let K be a number field. Let $\mathcal{O}$ be an order in K. Fix a prime p, and iterate the following process:

(i) Compute all elements of $\mathcal{O}$ such that $x^n \in p\mathcal{O}$ for some $n \geq 1$. This is an ideal of $\mathcal{O}$; call it X.

(ii) Replace $\mathcal{O}$ by $\mathcal{O}' = \{y \in K : yX \subset X\}$.

Prove that this stops after a finite number of steps. Prove that if we perform it for every prime dividing disc($\mathcal{O}$), the end result is the full ring of integers of K.

6. Let K be a number field and $\mathcal{O}$ the ring of integers in K. Prove that the function on $\mathcal{O}$-ideals defined by $N(I) :=$ size of $\mathcal{O}/I$ is multiplicative, i.e. for two ideals I, J we have $N(I) \cdot N(J) = N(IJ)$. Moreover, prove that for $\alpha \in \mathcal{O}$ we have

$$N((\alpha)) = |\text{Norm}_{K/\mathbf{Q}}\alpha|.$$

7. Let $K = \mathbf{Q}(\alpha)$ where $\alpha^3 = \alpha + 1$. Factor the ideal generated by $5 + 4\alpha + 3\alpha^2$ into prime ideals.

8. Let $K = \mathbf{Q}(\alpha)$ where $\alpha^3 = 3\alpha + 1$.

(i) Compute a few Frobenius elements and make a conjecture about the pattern.

(ii) Prove your conjecture.

(Hint for (ii): in fact $K \subset \mathbf{Q}(\zeta_9)$; figure out what goes in in $\mathbf{Q}(\zeta_9)$. Similar reasoning applied to $\mathbf{Q}(\sqrt{d})$ leads to quadratic reciprocity.)