Math 122, Midterm, due Friday 5pm

- You should work by yourself.
- You may consult the book or course notes, but no other sources.
- You are also welcome to consult me or the TA by email or at office hours.
- For all questions, liberal partial credit will be awarded, e.g. for working out any specific example correctly, for working out a special case of the general result, or by stating any ideas you have towards a solution.
- All vector spaces are finite-dimensional, all representations are on finite-dimensional vector spaces.

1. Let $G$ be a group (not necessarily finite) and $F$ a field.
   (a) Let $V$ be a $G$-representation. Give an example to show that $V$ may not be isomorphic to a direct sum of irreducible representations.
   (b) Suppose $V$ is isomorphic to a direct sum of irreducible representations. Let $X \subset V$ be any subrepresentation. Prove that there exists a homomorphism of $G$-representations $\pi : V \to V$ with image $X$.

   Hint for (b): first construct a $G$-subrepresentation $Y \subset V$ so that $V$ is the internal direct sum of $X$ and $Y$.

2. Let $G$ be an abelian group (not necessarily finite).
   (a) Explain why every irreducible representation of $G$ over $\mathbb{C}$ is 1-dimensional.
   (b) Deduce that, if $\rho : G \to \text{GL}(V)$ is a representation of $G$ on a complex vector space, there exists a basis $e_1, \ldots, e_n$ for $V$, with respect to which $G$ acts by upper triangular matrices.

   Hint for (b): You may wish to use the concept of “quotient representation”: if $W \subset V$ is a subrepresentation, then $G$ acts on the quotient $V/W$ via $g : v + W \mapsto \rho(g)v + W$.

3. Let $G$ be a finite group and $g \in G$ an element of order $N$ (i.e. $g^N = e$ and $N$ is the smallest such positive integer).
   (a) What is the characteristic polynomial of $g$ acting on the regular representation (i.e., action of $g$ by left multiplication on $CG$?)
   (b) Let $z \in \mathbb{C}$ satisfy $z^N = 1$. Prove that there exists an irreducible representation $\rho : G \to \text{GL}(V)$ on a complex vector space $V$ so that $\rho(g)$ has $z$ as an eigenvalue.

4. For each irreducible representation $\rho : S_5 \to \text{GL}(n, \mathbb{C})$, determine whether or not the image of $\rho$ lies inside the subgroup $\text{SL}(n, \mathbb{C})$ of matrices with determinant 1. (There’s a character table on page 884 of the text book.)

   Hint: $g \mapsto \det(\rho(g))$ is a homomorphism $S_5 \to \mathbb{C}^\times$, i.e. a one-dimensional representation.

5. Let $p$ be a prime, $V = \{\text{functions } \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}\}$ and let $S, T \in \text{GL}(V)$ be defined by:

\[
(Sf)(x) = f(x + 1), \quad (Tf)(x) = f(x)e^{2\pi ix/p}.
\]

Prove that the subgroup $G$ of $\text{GL}(V)$ generated by $S, T$ is a finite group of order $p^2$. Compute the character $\chi_V$ of this representation of $G$, and prove it is an irreducible representation.

---

\[1\text{in other words: the smallest subgroup of } \text{GL}(V) \text{ that contains } S \text{ and } T.\]