Problem Set 7 Solutions

Math 120

4.5.4 As D_{12} has order 12, its Sylow 2-subgroups all have order 4. By Sylow's theorem, we know these groups are pairwise conjugate, so we need only find one Sylow 2-subgroup and find all its conjugates. $\langle r^3, s\rangle$ is one such subgroup. This subgroup is clearly invariant under conjugation by powers of r, but conjugation by sr gives us the subgroup $\langle sr^2, r^3\rangle$ and conjugating by sr^2 gives us the subgroup $\langle sr^4, r^3\rangle$. By Sylow's theorem, we know there are n_2 Sylow 2-subgroups where $n_2 \equiv 1 \mod 2$ and $n_2|3$, so since we have already found three such subgroups, we have found all Sylow 2-subgroups. Similarly, D_{12} has n_3 Sylow 3-subgroups where $n_3 \equiv 1 \mod 3$ and $n_3|4$, so there are either 1 or 4 such subgroups. $\langle r^2\rangle$ is one. But this subgroup is normal, so it is the unique Sylow 3-subgroup of D_{12}.

$S_3 \times S_3$ has order 36, so its Sylow 2-subgroup has order 4. It is easy to see that every subgroup of the form $\langle((a\ b), 1), (1, (c\ d))\rangle$ is a subgroup of order four, and there are nine such subgroups. But Sylow's theorem tells us that there can be at most nine such subgroups, so these account for all of them. We also have from Sylow's theorem that that the number of Sylow 3-subgroups is 1 mod 3 and divides 4. One such subgroup is $\langle((1\ 2\ 3), 1), (1, (1\ 2\ 3))\rangle$. But this subgroup is easily checked to be normal (recall that for any $\sigma \in S_3$, $\sigma(1\ 2\ 3)\sigma^{-1} = (\sigma(1)\ \sigma(2)\ \sigma(3))$), so it is the unique Sylow 3-subgroup of $S_3 \times S_3$.

4.5.6 The group A_4 has order 12, so its Sylow 3-subgroups have order 3, and there are either 1 or 4 of them. Every group of order 3 is cyclic, so it is easy to write down four such subgroups: $\langle(1\ 2\ 3)\rangle$, $\langle(1\ 2\ 4)\rangle$, $\langle(1\ 3\ 4)\rangle$, and $\langle(2\ 3\ 4)\rangle$. Next note that the number of Sylow 3-subgroups in S_4 is 1 mod 3 and divides 8, and so there are either 1 or 4 such subgroups. But we have already found four such subgroups, so these account for all Sylow 3-subgroups in S_4.

4.5.15 The prime factorization of 351 is $3^3 \cdot 13$, so Sylow's theorem tells us that $n_3 = 1$ or 13 while $n_{13} = 1$ or 27. If a group G of order 351 does not have a normal Sylow p-subgroup, then $n_3 = 13$ and $n_{13} = 27$. Since 13 is prime, none of the 27 Sylow 13-subgroups can have nontrivial intersection, so G would have $(13-1)\cdot 27 = 324$ elements of order 13. This leaves only 27 elements left in the group, all of which must lie in a single Sylow 3-subgroup, contrary to assumption. This contradiction proves that G must have a normal Sylow p-subgroup.

4.5.18 This is a straightforward application of Sylow's theorem. $200 = 5^2 \cdot 2^3$, so $n_5 \equiv 1 \mod 5$ and $n_5|8$. These two conditions imply $n_5 = 1$, so the Sylow 5-subgroup must be normal.

4.5.33 Suppose $|G| = p^\alpha m$. First note that P contains every element of order p^β where $1 \leq \beta \leq \alpha$, for if $x \in G$ has order p^β, then $\langle x\rangle$ is a p-subgroup of G and thus, by Sylow's theorem, contained in a conjugate of P. Since P equals all of its conjugates, $\langle x\rangle \leq P \Rightarrow x \in P$. Now suppose Q is a Sylow p-subgroup of H. Then every element in Q has order p^β for some β by Lagrange's theorem, hence every element in Q lies in P. Since $Q \leq H$, we have $Q \leq P \cap H \Rightarrow Q = P \cap H$ since $P \cap H$ is easily seen to be a Sylow p-subgroup of H as it contains all elements in H of order p^β.

4.6.1 Suppose H is a proper subgroup of A_n of index $< n$, where $n \geq 5$. Then if A_n/H is the set of cosets of H, consider the permutation representation $\pi : A_n \to S_{A_n/H}$. Since $|A_n : H| < n$, we have $|S_{A_n/H}| \leq (n-1)! < n!/2 = |A_n|$. Moreover, the first isomorphism theorem gives us that $|A_n : \ker \pi| = |\text{image } \pi|$ which implies that $\ker \pi$ is a nontrivial proper normal subgroup of A_n contradicting the simplicity of this group. This contradiction proves that A_n has no proper subgroup of index $< n$.

4.6.2 Suppose $H \trianglelefteq S_n$ where $n \geq 5$. Then one easily checks that $H \cap A_n \trianglelefteq A_n$, so either $H \cap A_n = A_n$ or $H \cap A_n = 1$. If the former is true, then $A_n \leq H$, and hence $H = S_n$ or $H = A_n$ since A_n is an index 2 subgroup of S_n. If the latter is true, then the only non-identity elements in H are odd permutations. The product of two odd permutations is an even permutation, and so this would imply that H is either the trivial group or a group generated by a single odd permutation of order 2. We argue that the latter cannot be normal in S_n and therefore that the only normal subgroups of S_n are $1, A_n$, and S_n. Let $H = \langle \sigma \rangle$ where σ is an odd permutation of order 2. Any odd permutation of order 2 is a product of an odd number of disjoint transpositions. Suppose $(a\ b)$ is one of the transpositions in the cycle decomposition of σ. If $a \neq c \neq b$, then $(b\ c)\sigma(b\ c)^{-1}$ contains the cycle $(a\ c)$ in its cycle decomposition, while σ does not. Therefore $H \neq (b\ c)H(b\ c)^{-1}$, and so H is not normal, proving the assertion.

4.6.4 First note that if $n = 3$, $A_n = \{1, (1\ 2\ 3), (1\ 3\ 2)\}$ so the claim is obvious. Now suppose $n > 3$. Each element of A_n is a product of an even number of transpositions. Given $\sigma \in A_n$, write σ as such a product. The first two transpositions in this product either have the form $(a\ b)(c\ d)$ or $(a\ b)(a\ c)$ where a, b, c,and d are all distinct. The former can be writen as $(a\ b\ c)(b\ c\ d)$ while the latter can be written as $(a\ c\ b)$. We may similarly write the second pair of transpositions in terms of 3-cycles, and the third, and so on. Hence σ may be written as a product of 3-cycles, and A_n is therefore a subgroup of the group generated by 3-cycles. But every 3-cycle is an even permutation, and so these two groups are equal, proving the claim.