1. Math 120: practice final exam

- In questions with (a) and (b) parts, you may assume the result of part (a) when working on part (b), even if you couldn't solve (a).
- If you have an idea for a proof but are missing some steps, describe the idea *and* explain what is missing. I will give very liberal partial credit in this situation, but not for false "proofs."

(1) (a) Define a normal subgroup. Describe a normal subgroup of $\mathrm{GL}(3, \mathbb{R})$, besides the trivial group and the whole group.
(b) Given two normal subgroups H, K of a group G with $H \cap K = \{e\}$, prove that elements of H and K commute.

(2) (a) State precisely what it means that A_n is *simple* for $n \geq 5$. Prove that a group of prime order is simple.
(b) Prove that the only normal subgroup of S_n is A_n for $n \geq 5$.

(3) (a) How many rotations preserve a cube in $\mathbf{R}^3$? Sketch a proof.
(b) Suppose that G acts transitively on the finite set X. Show that there exists $g \in G$ which is a *derangement*: it does not fix any element of x, i.e. $g.x \neq x$ for all $x \in X$. (Hint: For each $x \in X$, let G_x be the stabilizer of x. Note that the identity is contained in every G_x. Show that $\bigcup G_x$ has size less than $|G|$).

(4) (a) Define precisely what it means for a group G to be generated by a subset $A \subset G$.
(b) Let G be a finite group that has more than one hundred subgroups of index 3. Prove that G cannot be generated by two elements. (Hint: count homomorphisms from G to S_3).

(5) (a) Let $\alpha = (123)(456) \in S_7$. Describe the conjugacy class of α and the centralizer of α.
(b) Prove that there is no finite group with just two conjugacy classes.

(6) Let G be a group of order 56. In this question, you may use Sylow's theorem but you should precisely state any part of it which you use.
(a) Prove that G has either one or eight Sylow 7-subgroups, and has either one or seven Sylow 2-subgroups.
(b) Suppose that there are eight Sylow 7-subgroups. Show that their union has exactly 48 elements, besides the identity. Conclude that there is a unique Sylow 2-subgroup.

(7) (a) Let R be a finite integral domain. Prove that R is a field.
(b) Prove that 11 is an irreducible element of $\mathbf{Z}[i]$.
(c) Show that $\mathbf{Z}[i]/(11)$ is a field.

(8) (a) Define *ideal* in the context of rings. Give an example of a subring that is not an ideal.
(b) Describe all ideals in the ring $\mathbf{Z}/60\mathbf{Z}$. Which ones are prime?

(9) (a) Let $f = x^2 + ax + b$ be a quadratic polynomial without real roots. Prove that $\mathbf{R}[x]/(f)$ is isomorphic to the ring of complex numbers. (Hint: let z be a complex root of f, and send x to z).
(b) Let I be any ideal of $\mathbf{R}[x]$. Prove that the quotient ring $\mathbf{R}[x]/I$ is isomorphic to a product of copies of $\mathbf{R}$ and $\mathbf{C}$.
(Hint: You may use the fact that the irreducibles in $\mathbf{R}[x]$ are linear polynomials and quadratic polynomials without real roots. Also, $\mathbf{R}[x]$ is a Euclidean domain. Use the Chinese remainder theorem.)