OPEN PROBLEMS (gathered by N. Berger, A. Bufetov, C. Demeter and N. Frantzikinakis)

I. Assani. Let $(X, \mathcal{B}, \mu)$ be a finite measure space, T_i be measure preserving transformations acting on X, and f_i be bounded functions, $i = 1, \ldots, 2^k - 1$. The averages along the cubes for three terms $(k = 2)$ and seven terms $(k = 3)$ are defined as follows:

$$M_N(f_1, f_2, f_3)(x) = \frac{1}{N^2} \sum_{m,n=1}^{N} f_1(T_1^m x) \cdot f_2(T_2^n x) \cdot f_3(T_3^{m+n} x)$$

and

$$\begin{aligned} M_N(f_1, f_2, \ldots, f_7)(x) = &\frac{1}{N^3} \sum_{m,n,p=1}^{N} f_1(T_1^m x) \cdot f_2(T_2^n x) \cdot f_3(T_3^p x) \cdot \\ &f_4(T_4^{m+n} x) \cdot f_5(T_5^{m+p} x) \cdot f_6(T_6^{n+p} x) \cdot f_7(T_7^{m+n+p} x). \end{aligned}$$

Similarly, we define the averages along the cubes for $2^k - 1$ terms and we denote them by $M_N(f_1, f_2, \ldots, f_{2^k-1})$. The averages along cubes for a single transformation (i.e. $T_i = T$ for all i) were introduced by V. Bergelson who proved L^2 convergence for $k = 2$. His result was extended by B. Host and B. Kra who proved L^2 convergence for $k > 2$ (for a single transformation). I. Assani proved a.e. convergence in this case.

Problem 1. Do the averages $M_N(f_1, f_2, \ldots, f_{2^k-1})$ converge in L^2 and a.e. for every $k \in \mathbb{N}$?

Assuming that the transformations T_i are ergodic and commute, I. Assani remarks that the answer is yes for every k, and without further assumptions the answer is yes for $k = 2$. If the transformations are weak mixing (not necessarily commuting) then the limit (a.e. and in L^2) exist and equals the product of the integrals.

Problem 2. Find characteristic factors for L^2 and a.e. convergence for the averages $M_N(f_1, f_2, \ldots, f_{2^k-1})$.

J. Campbell. For a sequence $\{T_n\}$ of operators and $n_1 < n_2 < \ldots < n_k < \ldots$ define the oscillation operator

$$O_2(T_n, f, \{n_k\})(x) := \left(\sum_{k=1}^{\infty} \sup_{n_k < n \le n_{k+1}} |T_{n_k} f(x) - T_n f(x)|^2 \right)^{\frac{1}{2}},$$

and the p-variation operator

$$V_p(T_n, f)(x) = \sup_{\{n_k\}} \left(\sum_k |T_{n_k} f(x) - T_{n_{k+1}} f(x)|^p \right)^{\frac{1}{p}},$$

$p > 2.$

An easy exercise shows that if $O_2(T_n, f)(x) < \infty \; a.e. \; x$ for each $\{n_k\}$ then $\lim_{n \to \infty} T_n f(x)$ exists a.e.

Set now

$$T_n^\theta f(x) = \sum_{0 < |k| \le n} \frac{e^{ik\theta} f(T^k x)}{k}$$

for each $\theta \in [0, 1)$ and f measurable in the dynamical system (X, Σ, m, T).

Problem 3. The fundamental question is whether it is true that there exists a Wiener-Wintner result for these averages, i.e. whether for each $f \in L^\infty(X)$ there exists $X_0 \subset X$ with $m(X_0) = 1$ such that $\lim_{n\to\infty} T_n^\theta f(x)$ exists for each $x \in X_0$ and $\theta \in [0,1)$.

Problem 4. If one wants to use harmonic analysis to answer this question, one approach is to prove an oscillation inequality for the continuous model and then transfer it back to the ergodic theory setting. This amounts to proving that for each decreasing sequence $\{t_k\}$ of positive real numbers, one has

$$\| \sup_{\theta\in[0,1)} \left(\sum_k \sup_{t_{k+1}\le t<t_k} \left| \int_{t_{k+1}\le|y|<t} \frac{f(x-y)}{y} e^{iy\theta} dy \right| \right)^{\frac{1}{2}} \|_{2,\infty} \lesssim \|f\|_2,$$

for $f \in L^2(\mathbb{R})$.

Denote also for $f \in L^1([0,2\pi))$

$$C_n f(x) = \sum_{|j|\le n} \hat{f}(j) e^{ijx}$$

Problem 5. Is it true that

$$V_p(C_n, f) : L^2 \to L^2$$

boundedly?

H. Furstenberg. A system (X,σ,τ) is a compact metric space X together with two commuting continuous transformations σ, τ acting on X. A system (X,σ,τ) satisfies the **dimension subadditivity** property if for every $x \in X$ we have

$$\dim \overline{\{\sigma^m\tau^n x\}}_{m,n\in\mathbb{N}} \le \dim \overline{\{\sigma^m x\}}_{m\in\mathbb{N}} + \dim \overline{\{\tau^n x\}}_{n\in\mathbb{N}}$$

where $\dim$ denotes the Hausdorff dimension. It satisfies the **transversality** property if whenever $A, B \subset X$ are closed such that $\sigma A \subset A$, $\tau B \subset B$, we have

$$\dim(A\cap B) \le \max\{\dim A + \dim B - \dim X, 0\}.$$

Problem 6. Let $n \in \mathbb{N}$ and σ, τ be expanding (continuous) homomorphisms of $\mathbb{T}^n$ with the standard metric such that every orbit $\{\sigma^m\tau^n x\}_{m,n\in\mathbb{N}}$ is either finite or dense, $x \in \mathbb{T}^n$. Does the system (X,σ,τ) satisfy the dimension subadditivity and/or the transversality property?

The assumption that both σ and τ are expanding is necessary as one can see by choosing $\sigma = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\tau = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$; then it can be shown that the system $(\mathbb{T}^2,\sigma,\tau)$ does not satisfy the dimension subadditivity or the transversality property.

Problem 7. Suppose that Λ is a finite additive group with the discrete metric. On the sequence space $X = \Lambda^{\mathbb{N}}$ with the product metric we let σ be the shift transformation and $\tau = 1 + \sigma$ (i.e. $(\tau x)_i = x_i + x_{i+1}$). Does the system (X,σ,τ) satisfy the dimension subadditivity property?

Note that this system doesn't satisfy the transversality property because there are many infinite closed subsets invariant under both transformations.

S. Gunturk. [Invariant sets of a class of piecewise affine maps on the Euclidean space]

Let **L** be the $n \times n$ lower triangular matrix of 1's, i.e.,

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 1 & \cdots & \cdots & 1 \end{pmatrix},$$

and $\mathbf{e}$ be the n dimensional vector $(1, \cdots, 1)^\top$.

Let x be a real number, $\Pi = \{\Omega_1, \ldots, \Omega_K\}$ be a partition of $\mathbb{R}^n$ into Lebesgue measurable sets, and $\mathcal{D} = \{\mathbf{d}_1, \ldots, \mathbf{d}_K\}$ be a set of vectors in $\mathbb{Z}^n$. Given the triple $(x, \Pi, \mathcal{D})$, we define a piecewise affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T(\mathbf{u}) = \mathbf{L}\mathbf{u} + x\mathbf{e} + \mathbf{d}_i, \text{ if } \mathbf{u} \in \Omega_i.$$

We say that T is *stable* if there exists a bounded set $A \subset \mathbb{R}^n$ of positive Lebesgue measure such that $T(A) \subset A$. It is easy to see that T is stable if and only if there exist bounded sets B and C in $\mathbb{R}^n$ of positive measure such that

$$\bigcup_{k=0}^{\infty} T^k(B) \subset C.$$

We would like to understand the invariant sets of the map T, given the triple $(x, \Pi, \mathcal{D})$. The following theorem is not difficult to prove:

Theorem[1] Let x be an irrational number and T be stable. Then there exist a finite and non-empty collection of disjoint sets $\tau_1, \ldots, \tau_N \subset \mathbb{R}^n$ such that

(a) each τ_i tiles $\mathbb{R}^n$ by $\mathbb{Z}^n$ translations, i.e., the collection

$$\left\{\tau_i + \mathbf{m} \ : \ \mathbf{m} \in \mathbb{Z}^n\right\}$$

is a partition of $\mathbb{R}^n$, and

(b) if we let $\Gamma = \bigcup \tau_i$, then $T(\Gamma) = \Gamma$. Also, we have $\Gamma = \bigcap_{k=0}^{\infty} T^k(A)$.

Notes.

(1) Note that the affine map $\mathbf{u} \mapsto \mathbf{L}\mathbf{u} + x\mathbf{e}$ is always unstable in all dimensions. The role of the partition Π and the corresponding set of translations $\mathcal{D}$ is to overcome this inherent instability.

(2) Perhaps the simplest stable maps T are those given (implicitly) by $T(\mathbf{u}) = \mathbf{L}\mathbf{u} + x\mathbf{e} \pmod 1$. Here, we have the invariant set $\Gamma = [0,1)^n$. (In reality, the corresponding partition Π would have infinitely many sets, but one can always reduce it to a finite partition by restricting the $\mathbf{d}_i$ to those produced by $\mathbf{u} \in [0,1)^n$. In general, this requires $K = 2^n$.)

(3) The most interesting case is when $K = 2$ regardless of the dimension n, i.e., when there are only two sets in the partition Π. This is the most challenging setup for finding stable maps T, though it is known that stable maps exist in all dimensions (for values of x in an interval).

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[1]For a proof, see S. Güntürk, N.T. Thao, "Ergodic Dynamics in Sigma-Delta Quantization: Tiling Invariant Sets and Spectral Analysis of Error," Advances in Applied Mathematics, in press. Available at `http://www.cims.nyu.edu/~gunturk/research.html`

(4) In one dimension, there is a strong link to interval exchange transformations. Higher dimensional versions are not much studied.

Some open problems.

Problem 8. For which $(x, \Pi, \mathcal{D})$ is T stable?
(Here, it would be desirable to have a characterization, or a set of sufficient or necessary conditions that can be checked, preferably via a finite algorithmic test.)

Problem 9. Determine the (maximal) invariant set Γ as an explicit function of the parameters of T (i.e., $(x, \Pi, \mathcal{D})$).

Problem 10. If this is not possible, predict the size of Γ and/or the number of tiles N in Γ. (Here, the single tile case $N = 1$ is particularly important as then the dynamics of T within Γ is isomorphic to the skew translation $\mathbf{u} \mapsto \mathbf{L}\mathbf{u} + x\mathbf{e} \pmod 1$ on $[0, 1)^n$, which is uniquely ergodic for irrational values of x.)

Problem 11. What is the regularity of Γ for a given map T?
(For instance, determine the Hausdorff dimension of $\partial\Gamma$. These results are useful in deriving quantitative results for the convergence of ergodic averages via the standard machinery of Weyl sums.)
Tiling invariant sets are observed even when x is rational, however the theorem mentioned above is unable to explain this. Explain the tiling phenomenon without using ergodic theory arguments.

J. P. Kahane. [Genericity and Prevalence]

Given a class of functions, if we say that a property holds *in general*, what does it mean?

We restrict ourselves to classes of functions that are Fréchet spaces. A "property" can be identified with a subset of the space.

Since a Fréchet space is a Baire space, we have the notion of *generic property*, mbelongs to this class or this intersection. eaning that it holds on a countable intersection of dense open sets. Instead of *generic* we also say *quasi sure.*

There is also the notion of *almost sure*, when we equip the Frechet space with a probability measure (we always assume that it is carried by a σ-compact set in the space).

When the Fréchet space is given, the notion of generic, or quasi sure, is well defined, but the notion of almost sure depends on the choice of a probability.

However, using the group structure of the space, we'll say that a property is *prevalent* if it is almost sure for **some** probability measure and **all** its translates.

The notion was introduced by Christiansen in 1972, and was developed by Hunt, Sauer, and Yorke in 1992 (Bull. AMS 27, 217–238; see also Bull. AMS 28, 306–307). I learned it from Stéphane Jaffard and his student Aurelia Fraysse, who have a series of examples in various spaces, where the properties they consider (multifractal formalism, failure of analytic continuation) are both generic and prevalent (2005). A simple example in $C(\mathbb{R})$ is the nowhere-differentiability, known to be generic and also prevalent, using the Wiener measure (Holicky and Zagicek, Acta Univ. Carol 41 (2000), 7–11).

Our general question can be split into two parts:

Problem 12. Give other examples of spaces and properties both generic and prevalent.

Problem 13. Give spaces and properties which are as different as possible from the generic and from the prevalent point of view.

Here is a contribution for question 2): The Fréchet space is the real space $C(\mathbb{R})$, and we consider a Cantor set E in $\mathbb{R}$ and a continuous probability measure μ on $\mathbb{R}$. We look at properties of the images of E and of μ by $f \in C(\mathbb{R})$, say, $F = f(E)$ and $\nu = \mu \circ f^{-1}$. Generically, F is a Kronecker set, meaning a Cantor set such that each function continuous on F whose absolute value is 1, can be approximated uniformly by imaginary exponentials. It is a thin set in most aspects of harmonic analysis (R. Kaufman, 1967).

Using this fact, one sees that ν is generically a singular measure, and moreover

$$\limsup_{n\to\infty} |\hat{\nu}(n)| = 1.$$

The prevalent properties are quite different. Prevalently, F has non-empty interior (question: is it a union of intervals?). Prevalently, ν is absolutely continuous, its density is C^∞ and moreover, given any non quasi-analytic class of infinitely differentiable functions (or a countable intersection of such classes), its density belongs to this class or this intersection.

Y. Katznelson. A set $\Lambda \subset \mathbb{N}$ is a set of recurrence for the system (X, d, T) if for every open set $U \subset X$ there exists $n \in \Lambda$ such that $U \cap T^{-n}U$ is nonempty. Λ is a set of **topological recurrence** if it is a set of recurrence for every minimal topological system, and a set of **Bohr recurrence** if it is a set of recurrence for every translation on a finite dimensional torus (with the standard metric).

Problem 14. If $\Lambda \subset \mathbb{N}$ is a set of Bohr recurrence is it necessarily a set of topological recurrence?

For background see Y. Katznelson, *Chromatic Numbers of Caley Graphs on $\mathbb{Z}$ and recurrence*, Combinatorica, 21, No 2, 211-219, also available at

`http://math.stanford.edu/~katznel`

B. Kra.

Problem 15. Let $(X, \mathcal{B}, \mu)$ be a finite measure space and let $T_1, \dots, T_k$ be ergodic commuting measure preserving transformations acting on X. If $\varepsilon > 0$ when is the set

$$\{n \in \mathbb{N} : \ \mu(A \cap T_1^{-n}A \cap \dots \cap T_k^{-n}A) > \mu(A)^{k+1} - \varepsilon\}$$

syndetic?

If $T_1 = T, T_2 = T^2, \dots, T_k = T^k$, V. Bergelson, B. Host and B. Kra showed that the set is always syndetic for $k \leq 3$ and gave examples where it is empty for $k = 4$.

Problem 16. Let $(X, \mathcal{B}, \mu)$ be a finite measure space, $T_1, \dots, T_k$ be commuting measure preserving transformations acting on X, and $p_1, \dots, p_k$ be linearly independent integer valued polynomials with zero constant term. If $\varepsilon > 0$ when is the set

$$\{n \in \mathbb{N} : \ \mu(A \cap T_1^{-p_1(n)}A \cap \dots \cap T_k^{-p_k(n)}A) > \mu(A)^{k+1} - \varepsilon\}$$

syndetic?

For a single transformation (i.e. $T_i = T$ for all i) N. Frantzikinakis and B. Kra showed that the set is syndetic for every $k \in \mathbb{N}$.

E. Lesigne. Two measure-preserving dynamical systems (X, T, μ), (Y, S, ν) are said to be *weakly disjoint* if, given $f \in L_2(X, \mu)$, $g \in L_2(Y, \nu)$, there exist sets $A \subset X$ and $B \subset Y$, $\mu(A) = 1$, $\nu(B) = 1$, such that for any $x \in A$, $y \in B$, the sequence

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) g(S^n y)$$

converges as $N \to \infty$.

We recall the following properties: disjointness implies weak disjointness; a dynamical system with discrete spectrum is weakly disjoint from any other system; the Chacon map is weakly disjoint from any other system (in particular, from itself); the Morse system is weakly disjoint from any ergodic system; two systems of positive entropy are never weakly disjoint.

Problem 17. Assume that a measure-preserving dynamical system is weakly disjoint from any ergodic system. Does it follow that it is disjoint from any measure-preserving dynamical system?

A subset $E \subset \mathbb{Z}$ is called a *set of strong recurrence* for a measure-preserving dynamical system (X, μ, T) if for an arbitrary subset $A \subset X$ of positive measure, there exists $\varepsilon > 0$ such that $\mu(A \cap T^{-n} A) > \varepsilon$ for infinitely many $n \in E$.

Problem 18. Let E be a set of strong recurrence for an arbitrary ergodic system. Does it follow that E is a set of strong recurrence for any measure-preserving dynamical system.

A. Naor. Let X, Y be metric spaces. Assume that there exists a constant C such that for any subset $Z \subset X$, any $K > 0$, and any Lipschitz map $F : Z \to Y$ with Lipshitz constant K, one can find an extension $\mathbf{F} : X \to Y$, agreeing with F on Z and having Lipschitz constant at most CK. The smallest such C is denoted by $e(X, Y)$ (we set $e(X, Y) = \infty$ if no such constant exists).

The classical theorem of Kirszbraun (1934) says that $e(H_1, H_2) = 1$ for two Hilbert spaces H_1, H_2: in other words, a Lipschitz map from a subset of a Hilbert space into another Hilbert space can be extended to a global map between these spaces without any increase in the Lipschitz constant.

In 1992, Keith Ball showed that $e(L_2, L_p)$ is finite for $1 < p < 2$.

Problem 19. Is $e(L_2, L_1)$ finite or infinite?

For maps into Banach spaces, a different approach to the extension problem was suggested by Lee and Naor (2003). Given a metric space X, assume that there exists a constant C such that for any metric space Y, containing X, any Banach space Z, and any Lipschitz map $f : X \to Z$ with Lipschitz constant K, there exists an extension $\mathbf{F} : Y \to Z$, agreeing with f on X and having Lipschitz constant at most CK. The smallest such C is called the *absolute extendability constant* of X and denoted by $ae(X)$ (again, we set $ae(X) = \infty$ if no such constant exists). The problem of estimating the absolute extendability constant is already interesting for finite metric spaces. It is known that there exist two constants C_1 and C_2 such that

$$C_1 \sqrt{\frac{\log n}{\log \log n}} \le \sup_{|X| \le n} ae(X) \le C_2 \frac{\log n}{\log \log n}.$$

Problem 20. What is the precise asymtotics in n of $\sup_{|X| \le n} ae(X)$?

D. Ornstein. Find an approach to KAM (i.e. a method to produce invariant curves and surfaces) that is: (i) elementary, (ii) unified, (iii) gives results that are optimal, (iv) relaxes the condition: small perturbation of a completely integrable system. This can be done in dimensions 1 and 2 (work partly joint with Y. Katznelson).

In dimension 1, given a diffeomorphism ψ of the circle the method gives the complete answer to the smoothness of the invariant measure (i.e. the function conjugating ψ to a rigid rotation) in terms of the smoothness of ψ and the rate of growth of the coefficients of the continued fraction expansion of the rotation number α. It gives the optimal results for any diophantine α (completing results of Herman and Yoccoz).

The main results in dimension 2 are:

(A) Let ψ be a measure preserving diffeomorphism of the disc and assume that the rotation number α of ψ restricted to the boundary is diophantine, and that ψ is smooth enough given α. Then the method produces invariant curves of optimal smoothness. For example if $\psi \in C^{3+\gamma}$ and the coefficients of the continued fraction expansion of α grow polynomially then there are invariant curves (filling a set of positive measure near the boundary) in $C^{2+\gamma}$. A recent result of Herman produces invariant curves in $C^{1+\gamma}$. He uses standard KAM where the loss in smoothness is not optimal because the curve and its invariant measure are produced together (and the invariant measure is less smooth than the invariant curve). Standard KAM can be applied because if the boundary is rotated rigidly then near the boundary we have a small perturbation of a completely integrable system. Conjugating to this situation requires more smoothness for ψ. Note that we could replace the circle by any invariant curve in $C^{1+\gamma}$.

Problem 21. Is there a 3-dimensional analog of (A)?

(B) We assume that ψ is a twist map of the disc or the cylinder. Then a necessary and sufficient condition for the existence of an invariant curve in C^β ($\beta > 2$) with rotation number α is: A "sufficiently long" finite orbit lies on a C^β curve. We get the optimal smoothness needed for ψ in terms of α and β and "sufficiently long" depends on α and β. The completely integrable system has disappeared (as in the Morse twist) but the ghost of a "small" perturbation appears in "sufficiently long". The main machinery of our method does not depend on being in dimension 1 or 2. We can get an analog of (B) for invariant 2-tori in a 3-torus.

Problem 22. Find and prove optimal smoothness conditions in this situation.

Y. Peres. [Projections of planar Cantor sets and a related Kakeya set] Let K_n be the product of two n-stage middle-half Cantor sets, and let $K = \cap K_n$ be the product of the two Cantor sets. For $\theta \in [0, \pi)$, let P_θ be the operator of projection at angle θ. It is well-known (Besicovitch) that

$$\lim_{n\to\infty} \int_0^\pi \mu(P_\theta(K_n))d\theta = \int_0^\pi \mu(P_\theta(K))d\theta = 0.$$

Problem 23. What is the rate of convergence? It is known (Peres, Solomyak) that

$$\frac{C}{n} < \int_0^\pi \mu(P_\theta(K_n))d\theta < \frac{c}{\log^*(n)}$$

for some constants C and c, where $\log^*(n)$ is defined to be the minimal k such that $n \leq \exp^{(k)}(1)$ and $\exp^{(k)}$ denotes the kth iterate of $\exp$. .

Note that this is related to a construction of Kakeya set, i.e. a compact set of area zero containing a unit interval in every direction: Let K be a middle half cantor set, and let

$K^{(1)}, K^{(2)} \subseteq \mathbb{R}^2$ be $K^{(1)} = K \times \{0\}$ and $K^{(2)} = \frac{1}{2}K \times \{1\}$. Take $C = \{\alpha x + (1 - \alpha y) | 0 < \alpha < 1, x \in K^{(1)}, y \in K^{(2)}\}$. Then C contains a unit interval in every direction within a range of 60 degrees, and the cuts $C \cap (\{\alpha\} \times \mathbb{R})$ are projections of K^2.

D. Rudolph. We start by defining an $\mathbb{R}^n$ or $\mathbb{Z}^n$ borel foliation of a Polish space X. This will be a countable collection of charts C_i which have the form $S_i \times B_{r_i}$ where S_i is Polish and B_{r_i} is a ball or box of radius r_i centered at $\vec{0}$ in $\mathbb{R}^n$ or $\mathbb{Z}^n$. With each chart C_i we have a borel injection $\phi_i : C_i \to X$, i.e.

(1) the sets $\phi_i(C_i)$ cover
(2) on intersections $I_{i,j} = \phi_i^{-1}(\phi_i(C_i) \cap \phi_j(C_j))$ restricted to to a leaf $(s \times B_{r_i}) \cap I_{i,j}$ will have the form
$$s \times (B_{r_i} \cap (f_{i,j,s}(B_{r_j}))),$$
where $f_{i,j,s}$ is an isometry of $\mathbb{R}^n$ or $\mathbb{Z}^n$
(3) for all $x \in X$, $x \in \phi_i(C_i)$ if one seeks to develope the leaf through x starting with $s \times B_{r_i}, \phi_i(s, \vec{v}) = x$, and extending through leaves of charts that intersect this leaf, one obtains a full copy of $\mathbb{R}^n$ or $\mathbb{Z}^n$.

We call the leaf through x, L_x. A natural example would be free borel actions of $\mathbb{R}^n$ or $\mathbb{Z}^n$ on a Polich space.

Now suppose μ is a borel probability measure on X. On any chart C_i we can consider $\mu_i = (\phi_i^{-1})_*(\mu)$, the pull back of μ to a chart. On a chart one can take the Rohklin decomposition of the measure $\mu_i = \int \mu_s d\mu_i(s)$. The measures μ_s are unique up to normalization. This means that on intersections of charts, the measures μ_i and $(\phi_i^{-1}\phi_j)_*\mu_j$ agree up to a normalization. This means, for μ a.e. x one can construct a measure μ_x obtained by extending the measures chart by chart with the correct normalization, to the full leaf L_x. For expliciteness we normalize so that for a.e. x $\mu_x(B_1(x)) = 1$, where $B_1(x)$ is the unit ball about x in L_x.

For any $f \in L^1(\mu)$ one can use the μ_x to compute leaf averages
$$A_{r,\mu}(x) = \frac{\int f d\mu_x}{\mu_x(B_r(x))}.$$

Problem 24. Is it true, at this level of generality that $A_{r,\mu}(x)$ converges in r a.s., or in alternatively in $L^1(\mu)$?

It is known that (1) the answer is yes for $\mathbb{R}$ or $\mathbb{Z}$ foliations, by a slight extension of the Hurewicz ergodic theorem. (2) For $\mathbb{R}^n$ or $\mathbb{Z}^n$ foliations one does have a general maximal inequality (these are joint work with E. Lindenstauss).

W. Schlag. Let $H_{\omega,\lambda}$ be the disctete quasi-periodic Schrödinger operator given by the skew-shift:
$$(H_{\omega,\lambda}\psi)_n = -\psi_{n+1} - \psi_{n-1} + \lambda v(T_\omega^n(x,y))\psi_n,$$
where $(\psi_n) \in l_2$, $v(x,y) = \cos(2\pi x)$ for each $(x,y) \in \mathbb{T}^2$ and $T_\omega(x,y) = (x+y, y+\omega)$ is the skew-shift on the 2 dimentional torus $\mathbb{T}^2$. For each eigenvalue E we denote by
$$A_j(x,y;E) = \begin{pmatrix} \lambda v(T_\omega^j(x,y)) - E & -1 \\ 1 & 0 \end{pmatrix},$$
$$M_n(x,y;E) = \prod_{j=n}^{1} A_j(x,y;E),$$
$$L_n(E) = \int_{\mathbb{T}^2} \frac{1}{n} \log \|M_n(x,y;E)\| dxdy$$

and $L(E) = \lim_{n\to\infty} L_n(E) = \inf L_n(E)$ denotes the Lyapunov exponent. Clearly $L(E) \geq 0$, for all E and λ. J. Bourgain, M. Goldstein and W. Schlag have proved that for each ϵ there exists a set $\Omega_\epsilon \subset \mathcal{T}$ with $\text{mes}[\mathcal{T} \setminus \Omega_\epsilon] < \epsilon$ and a large constant $\lambda_0(\epsilon)$ such that for each $\omega \in \Omega_\epsilon$ and $\lambda \geq \lambda_0$, the Lyapunov exponents $L(E)$ are strictly positive for all energies, for a.e. (x, y).

Problem 25. Is it true that one can replace $\lambda \geq \lambda_0$ with $\lambda > 0$ in the above?

This would contrast with the case of simple shift on $\mathcal{T}$, $T_\omega(x) = x + \omega$, where the result only holds for $\lambda > 2$.

Scott Sheffield. [Differentiability of infinity harmonic functions]

We begin with a $n \times n$-grid on the plane and a real-valued function $f(x, y)$, defined on the vertices of the grid.

The game of *tug of war* has two players and a moving point, which is placed at the origin in the beginning of the game. Each player moves the point into an adjacent vertex of the grid. When the point hits the boundary of the grid at a point (x_0, y_0), the first player collects $f(x_0, y_0)$ dollars.

Problem 26. Is it true that there exist $N \in \mathbb{N}$ and $\delta > 0$ such that if $n > N$ and $|f(x, y) - y| < \delta n$, then the optimal first move for the first player is up?

We can now consider the continuous version of the game. Here f is a continuous function on the boundary of the $n \times n$-grid, and the players are allowed to move the point at distance one in any direction. Fix $C > 0$. We say that the point goes "up" by a given move of the game if the vertical coordinate of the vector by which it is moved is at least C. It would be interesting to answer the question in this case also.

B. Solomyak. [Interior points of self-similar sets]

A nonempty compact set $E \subset \mathbb{R}$ is called *self-similar* if there exists $m \in \mathbb{Z}$, $m \geq 2$, $\lambda_1, \dots, \lambda_m \in \mathbb{R}$, $0 < \lambda_i < 1$, and $d_1, \dots, d_m \in \mathbb{R}$ such that

$$E = \cup_{i=1}^m f_i(E), \text{ where } f_i(x) = \lambda_i x + d_i.$$

We shall sometimes write

$$E = E(\lambda_1, \dots, \lambda_m; d_1, \dots, d_m).$$

Problem 27. Let E be a self-similar of positive Lebesgue measure. Does E have nonempty interior?

If $m = 2$, then it is easy to see that the answer is "yes"; we shall therefore assume $m \geq 3$ in what follows.

The *similarity dimension* of E is $\alpha > 0$ such that

$$\sum_{i=1}^m \lambda_i^\alpha = 1.$$

It is well-known that the Hausdorff dimension of E does not exceed its similarity dimension, so we are interested in the case $\alpha \geq 1$. If $\alpha = 1$, then the answer to Question 27 is "yes" (A. Schief). For $\alpha > 1$, it is known (Marstrand and Mattila), that, given $\lambda_1, \dots, \lambda_m$, for *almost every* m-tuple $d_1, \dots, d_m$ the corresponding self-similar set has positive Lebesgue measure. We thus have

Problem 28. Take a vector $(\lambda_1, \dots, \lambda_m)$, whose self-similarity dimension is greater than 1. Is it true that for almost every (with respect to Lebesgue) vector $(d_1, \dots, d_m)$, the set $E = E(\lambda_1, \dots, \lambda_m; d_1, \dots, d_m)$ has nonempty interior?

Questions 27 and 28 are already interesting for special families of self-similar sets. For example, let $\frac{1}{4} < \lambda < \frac{1}{2}$, denote by C_λ the standard middle $1-2\lambda$ Cantor set and consider its Cartesian square $K_\lambda = C_\lambda \times C_\lambda$. For $0 \leq \theta \leq 2\pi$, denote by p_θ the orthogonal projection onto a line with slope θ. Consider the family $p_\theta(K_\lambda)$. By Marstrand, for almost all θ, the Lebesgue measure of $p_\theta(K_\lambda)$ is positive. Is it also true that the interior of $p_\theta(K_\lambda)$ is nonempty for almost all θ? is there at least a single θ such that $p_\theta(K_\lambda)$ has positive measure but empty interior?

One may also ask what happens in higher dimensions. Recently, an example was found (joint with M. Csörnyei, T. Jordan, M. Pollicott, and D. Preiss) of a self-similar set in $\mathbb{R}^2$ with positive Lebesgue measure but empty interior.

B. Weiss. [Shanon Entropy of linear factors]

Problem 29. Let $\{X_n\}_{n=-\infty}^{\infty}$ be a stationary process in the reals such that $h(X_n) = \infty$. Assume further that the X_n-s are bounded. Let $0 \neq \{c_n\} \in \ell_1(\mathbb{Z})$ and let Y_n be the convolution

$$Y_n = \sum_{k=-\infty}^{\infty} Y_{n-k} c_k.$$

Is it true that $h(Y_n) = \infty$?

Problem 30. Suppose $\{X_{i,j}\}_{(i,j)\in\mathbb{Z}^2}$ is a stationary finite valued process. Let $0 \neq \{c_n\} \in \ell_1(\mathbb{Z}^2)$ and let $Y = c * X$. Is it true that $h(X) = h(Y)$?

M. Wierdl. Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be a non-increasing sequence satisfying $0 \leq \sigma_n \leq 1$, and $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of independent $0-1$ valued random variables with $P(X_n(\omega) = 1) = \sigma_n$. Given $\omega \in \Omega$ we construct the integer subset A^ω of $\mathbb{N}$ by taking $k \in A^\omega$ if and only if $X_k(\omega) = 1$. By writing the elements of A^ω in increasing order we get a sequence $\{a_n(\omega)\}_{n\in\mathbb{N}}$. M. Boshernitzan and J. Bourgain showed that if

$$\lim_{t\to\infty} \frac{w(t)}{\log t} = \infty \tag{1}$$

where $w(t) = \sum_{n\leq t} \sigma_n$ then ω-almost surely we have: in every measure preserving system the ”random averages”

$$\frac{1}{w(t)} \sum_{\substack{n\leq t\\ n\in A^\omega}} T^n f$$

converge in L^2 as $t \to \infty$. Moreover, they showed that ω-almost surely the set A^ω is a set of (single) recurrence.

Problem 31. Assuming that (1) holds, is it true that ω-almost surely we have: for every measure preserving system and bounded measurable functions f, g the multiple ergodic averages

$$\frac{1}{w(t)} \sum_{\substack{n\leq t\\ n\in A^\omega}} T^n f\, T^{2n} g$$

converge in L^2 as $t \to \infty$?

A set $\Lambda \subset \mathbb{N}$ is called a set of **double recurrence** if for every measure preserving system and measurable set A with positive measure there exists $n \in \Lambda$ such that $\mu(A \cap T^{-n}A \cap T^{-2n}A) > 0$.

Problem 32. Assuming that (1) holds, is it true that ω-almost surely the set A^ω is a set of double recurrence?

M. Wierdl notes that the answer to both questions is yes if $\sigma_n = 1/n^a$ for some $0 \leq a < 1/2$, and unknown if $a = 1/2$.