The Propagation of Singularities for the Wave Equation on Manifolds with Boundary

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June 13, 2008

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0 Introduction

Light propagates in straight lines in a homogeneous space and reflects/refracts obeying Snell's law when it hits a surface. For the case of hyperspaces, we get the usual law of incident and reflected rays enclosing an equal angle to the normal to the surface. From the theory of electromagnetic radiation we know that these optical phenomena are governed by a partial differential equation called the wave equation. Thus the study of geometric optics is highly related with the study of solutions to a partial differential equation.

Although solving the translation-invariant (i.e. constant coefficients) wave equation on a vector space can be done with elementary theories of partial differential equations, explicitly solving the wave equation with variable coefficients on an arbitrary manifold is complicated. Thus when we study partial differential equations on a manifold we are usually more interested in local behaviors of solutions than explicitly solving the equations. For the wave equation, the propagation of singularities is of our particular interest because it is highly related to the propagation of radiating solution waves. In other words, singularities of solutions of the wave equation follow geometric optics rays.

This paper is a research paper about the propagation of singularities on a manifold with boundary. The first part of the paper provides a review of preliminaries in distribution theory and the theory of differential manifolds. The second part then explores some advanced concepts in the theory of partial differential equations such as pseudodifferential operators and wavefront set and shows how the wavefront set describes the propagation of singularities of solutions to an arbitrary operator in the boundaryless setting. On the third part we will narrow down our focus to the wave operator in the boundary setting and state an analogous relationship between the wavefront set and the propagation of singularities of solutions to that in the boundaryless setting.

It should be emphasized that this paper is mostly influenced by a Stanford University professor Andras Vasy. We also emphasize this paper is a research paper. Therefore the concepts and the arguments are stated as concisely as possible. For this purpose the basic reference is to Hormander and Melrose-Sjostrand who proved the theorems in the boundaryless setting and the boundary setting, respectively. For the proof of the theorems however the paper only gives a sketch for important theorems so that the paper is written with the least amount of complexity.

1 Preliminaries

Before we start our discussion about the singularity propagation theorem, we first review elementary distribution theory the theory of differential manifolds.

1.1 A review of distribution theory

1.1.1 Test functions

In the theory of distributions, one must work consistently with smooth test functions.

Definition 1.1.1. Let X be an open set in $\mathbb{R}^n$. By $C_0^k(X)$ we denote the space of all $u \in C^k(X)$ with compact support. The elements of C_0^∞ are called test functions.

Here $C^k(X)$ is the conventional notation for the space of k times continuously differentialbe complex valued functions in X.

Test functions can be used to identify continuous functions, or even locally integrable functions.

Theorem 1.1.2 *If $f, g \in C(X)$ and*

$$\int f\phi \mathrm{d}x = \int g\phi \mathrm{d}x \qquad \phi \in C_0^\infty(X) \tag{1}$$

then $f = g$.

More generally, if f, g are locally integrable functions in X and (1.2) *is valid, then $f = g$ almost everywhere in X.*

The proof is based on the following lemma.

Lemma 1.1.3 *Let x_0 be an arbitrary point in $\mathbb{R}^n$. There exists a non-negative function $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi(x_0) > 0$ and support in the ball of radius δ with center at x_0.*

Proof. First we construct a non-negative function $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi(0) > 0$. Take $f(t) = \exp(-1/t), t > 0$ and $f(t) = 0, t \le 0$. It is easy to verify that f is in $C^\infty(\mathbb{R})$. Then ψ is defined by $\psi(x) = f(1 - |x|^2)$.

By translation and change of scales we obtain the non-negative C_0^∞ function

$$\phi(x) = \psi(\frac{x - x_0}{\delta})$$

which has the required properties.

Now we can prove Theorem 1.1.2.

Proof of Theorem 1.1.2. If $h = f - g$ we have

$$\int h\phi \mathrm{d}x = 0, \qquad \phi \in C_0^\infty(X) \tag{2}$$

We first consider the case $f, g \in C(X)$. Taking real and imaginary parts we may assume that h is real valued provided that ϕ is taken real valued. Suppose there exists a point x_0 with $h(x_0) \neq 0$. By Lemma 1.1.3, we can take $\phi \in C_0^\infty(X)$ non-negative with $\phi(x_0) \neq 0$ and support so close to x_0 that ϕh has a constant sign. However this contradicts (2), and we conclude that $h = 0$ identically as claimed.

Now assume that f, g are locally integrable functions on X and (1). It suffices to show that if h satisfies (2) then $h = 0$ almost everywhere. To do so we use Lebesgue's theorem stating that

$$\lim_{t\to 0} t^{-n}\int_{|x-y|<t} |h(x)-h(y)|\,\mathrm{d}y = 0$$

for almost every x. With $\phi \in C^\infty(X)$ having support in the unit ball and $\int \phi\,\mathrm{d}x = 1$, we can write for $x \in X$ and small t

$$\begin{aligned} h(x) &= \int h(x)\phi(\frac{x-y}{t})\frac{1}{t^n}\,\mathrm{d}y \\ &= \int (h(x)-h(y))\phi(\frac{x-y}{t})\frac{1}{t^n}\,\mathrm{d}y + \int h(y)\phi(\frac{x-y}{t})\frac{1}{t^n}\,\mathrm{d}y. \end{aligned}$$

The last integral vanishes by hypothesis and the preceding one tens to 0 with t for almost all x, which proves that $h(x) = 0$ almost everywhere.

Now we shall see how this identification of continuous functions or locally integrable functions via test functions enables us to accept non-classical solutions, or *weak solutions* of PDEs. Classical solutions of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{3}$$

or the wave equation (in two variables)

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = 0 \tag{4}$$

are twice continuously differentiable functions satisfying the equations everywhere. It is easily shown that uniform limits of classical solutions of the Laplace equation are classical solutoins. We can even show that all classical solutions are indeed smooth. On the other hand, the classical solutions of the wave equation are all functions of the form

$$v(x,y) = f(x+y) + g(x-y) \tag{5}$$

with twice continuously differentiable functions f and g, and they have as uniform limits all functions of the form (5) with f and g continuous. All functions of the form (5) with f and g continuous ought therefore to be recognized as solutions of (4) so the definition of a classical solution is too restrictive.

Let us now consider the corresponding inhomogeneous equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F \tag{6}$$

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = F \tag{7}$$

where F is a continuous function vanishing outside a bounded set. If F is continuously differentiable a solution of (7) is given by

$$v(x,y) = \iint\limits_{\eta-y+|x-\xi|<0} -F(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{8}$$

However, (8) defines a continuously differentiable function v even if F is just continuous. Clearly we must accept v as a solution of (7) in this case even if second order derivatives donot exist. Similarly (6) has the classical solution

$$u(x,y) = \frac{1}{4\pi}\iint F(\xi,\eta)\log((x-\xi)^2+(y-\eta)^2)\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{9}$$

provided that F is continuously differentiable. Again (9) defines a continuously differentaible function u even if F is just continuous, and we should be able to accept u as a solution of (6) also in that case.

The flaws of the classical solutions illustrated in the preceding examples are eleminated by the concept of *weak solution* which preceded distribution theory. The idea is to rewrite the equation considered in a form where the unkonwn function u is longer differentiated. Consider as an example the equation (7). If u is a classical solution it follows that

$$\iint \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 y}{\partial y^2}\right)\phi\,\mathrm{d}x\,\mathrm{d}y = \iint F\phi\,\mathrm{d}x\,\mathrm{d}y \tag{10}$$

for every continuous function ϕ vanishing outside a compact set. Conversely, if (10) is fulfilled for all such smooth (i.e. infinitely differentiable) ϕ then (7) is fulfilled. In fact, if (7) were not satisfied at a point (x_0, y_0) we could take ϕ non-negative and 0 outside a small neighborhood of (x_0, y_0) and conclude that (10) is not fulfilled either.

Note that if ϕ is smooth we can integrate by parts twice in the left-hand side of (1) which gives the equivalent formula

$$\iint u\left(\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2}\right)\mathrm{d}x\,\mathrm{d}y = \iint F\phi\,\mathrm{d}x\,\mathrm{d}y. \tag{11}$$

Summing up, if u is twice continuously differentiable then (7) is equivalent to the validity of (11) for all smooth functions ϕ vanishing outside a compact set. However, (11) has a meaning if u is just continuous, and one calls u a weak solution of (7) when (11) is valid for all such functions ϕ.

Observe that the space of "test functions" ϕ here and that in Definition 1.1.1 coincide. Indeed identification of u via (11) is valid because of Theorem 1.1.2.

1.1.2 Distributions

The classical derivative of weak solutions usually do not exist. We can accept weak solutions as solutions of PDEs only if they have well-defined derivatives. Accordingly the concept of weak solutions motivates the need for a more general definition of derivatives in the theory of partial differential equations. As a minimal extension of the space of continuous functions where differentiation is always possible, we define the space of distributions.

Definition 1.1.4. A distribution u in X is a linear form on $C_0^\infty(X)$ such that for every compact set $K \subset X$ there exist constant C and k such that

$$|u(\phi)| \leq C \sum_{|\alpha| \leq k} \sup |\partial^\alpha \phi|, \qquad \phi \in C_0^\infty(K). \tag{12}$$

The set of all distributions in X is denoted by $\mathcal{D}'(X)$. If the same integer k can be used in (12) for every K we say that u is of order $\leq k$, and we denote the set of such distributions by $\mathcal{D}'^k(X)$. Their union $\mathcal{D}'_F(X) = \cup \mathcal{D}'^k(X)$ is the space of distributions of finite order.

Note that, by Theorem 1.1.2, the space of continuous functions can be identified with a subspace of $\mathcal{D}'(X)$ by assigning to each continuous function f the distribution

$$C_0^\infty(X) \ni \phi \to \int f \phi \mathrm{d}x$$

which we also denote by f.

Definition 1.1.5. The *function of rapid decrease* $\mathcal{S}(\mathbb{R}^n)$ is the set of infinitely differentiable complex-valued functions ϕ on $\mathbb{R}^n$ such that for every $\alpha, \beta \in \mathbb{N}_0^n$

$$\|\phi\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\beta D^\alpha \phi(x)| < \infty.$$

The topological dual space of $\mathcal{S}(X)$, denoted by $\mathcal{S}'(X)$, is called the space of *tempered distributions.*

The norms on $\mathcal{S}(X)$ is given by

$$\|\phi\|_k = \max_{|\alpha|+|\beta| \leq k} \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \phi(x)| \tag{13}$$

In fact, the space $\mathcal{D}'(X)$ is an extension of $\mathcal{S}'(X)$ obtained by inductive limits. Many topological or analytical facts about tempered distributions are likely to hold for distributions. For our analysis, therefore, we sometimes study $\mathcal{S}'(X)$ (or $\mathcal{S}(X)$) first and generalize the results to $\mathcal{D}'(X)$ (or $C_0^\infty(X)$).

Now we define the derivatives of distributions as follows.

Definition 1.1.6. If $u \in \mathcal{D}'(X)$ we set

$$(\partial_k u)(\phi) = -u(\partial_k \phi), \qquad \phi \in C_0^\infty(X). \tag{14}$$

This definition indeed generalizes the definition of the classical derivatives. Suppose that u is a differentiable function of X. This simply means that the classical derivatives $\partial_k g$ of g exist and are continuous. Further $\partial_k g$ are identified as distributions as in (1.1.2), so we have

$$(\partial_k u)(\phi) = \int \partial_k u \, \phi \, \mathrm{d}x = \int -u \, \partial_k \phi \, \mathrm{d}x = -u(\partial_k \phi) \qquad \phi \in C_0^\infty(X)$$

where the second identity follows from the fact that ϕ is compactly supported.

Similarly we can define the multiplication by smooth functions.

Definition 1.1.7. If $u \in \mathcal{D}'(X)$ and $f \in C^\infty(X)$ we define

$$(fu)(\phi) = u(f\phi), \qquad \phi \in C_0^\infty(X). \tag{15}$$

It is clear that (13) and (14) define distributions $\partial_k u$ and fu.

For a multiindex $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_n)$, we write $|\alpha| = \sum_{j=1}^{n} \alpha_j$. If $p(x_1, x_2, \cdots, x_n)$ is a polynomial of total degree k in n variables, $p(x_1, x_2, \cdots, x_n) = \sum_{|\alpha| \le k} a_\alpha x^\alpha$, the partial differential operator $p(D) = \sum_{|\alpha| \le k} a_\alpha D^\alpha$ extends to $\mathcal{D}'(X)$ by the formula

$$(p(D)T)(\phi) = T(\sum_{|\alpha| \le k} (-1)^{|\alpha|} D^\alpha (a_\alpha \phi)).$$

Now we give a formal definition of weak solutions.

Definition 1.1.8. Let $p(D)$ be a partial differential operator associated with a polynomial $p(x_1, x_2, \cdots, x_n)$. If $T \in \mathcal{D}'(X)$ and $p(D)T = f$, then T is called a *weak solution* of the partial differential equation $p(D)u = f$.

In the theory of partial differential equations, we are mostly interested in local behaviors of solutions. Because weak solutions are defined as distributions, it is crucial to have a way to analyze local properties of distributions. The distribution theory developed such a concept called localizatoin.

Definition 1.1.9. If $Y \subset X \subset \mathbb{R}^n$ and $u \in \mathcal{D}'(X)$, we can restrict u to a distribution u_Y in Y by setting

$$u_Y(\phi) = u(\phi), \qquad \phi \in C_0^\infty(Y).$$

The fundamental theorem of localization tells that a distribution is determined by the restrictions to the sets in an open covering.

Theorem 1.1.10. **[4, Theorem 2.2.4]** *Let X_i, $i \in I$, be an arbitrary family of open sets in $\mathbb{R}^n$, and set $X = \cup X_i$. If $u_i \in \mathcal{D}'(X_i)$ and $u_i = u_j$ in $X_i \cap X_j$ for all $i, j \in I$, then there exists one and only one $u \in \mathcal{D}'(X)$ such that u_i is the restriction of u to X_i for every i.*

This theorem makes it natural to define the support of distributions.

Definition 1.1.11. If $u \in \mathcal{D}'(X)$ then the support of u, denoted supp u, is the set of points in X having no open neighborhood to which the restriction of u is 0.

Thus $X\backslash$ supp u is the open set of points having a neighborhood in which u vanishes, so u vanishes in $X\backslash$ supp u by Theorem 1.1.10, and $X\backslash$ supp u contains every open set where u vanishes. The following characterizatoin of supp u is usually found to be useful.

Proposition 1.1.12. *$x \notin$ supp u if and only if there exists a nonnegative function $\phi \in C_0^\infty(X)$ such that $\phi(x) \neq 0$ and $\phi u = 0$.*

Proof. Suppose that x is not in supp u. By definition, we can find a neighborhood Y of x to which the restriction of u is zero. Consider a nonnegative function $\phi \in C_0^\infty(Y) \subset C_0^\infty(X)$ which is identically 1 around x. Then we have $\phi(x) \neq 0$. We also note that we have $\phi u(\psi) = u(\phi\psi) = 0$ $\forall \psi \in C_0^\infty(Y)$ as the restriction of u to Y is zero, which means $\phi u = 0$ by definition. Conversely, we assume that there exists a nonnegative function $\phi \in C_0^\infty(X)$ with $\phi(x) \neq 0$ and $\phi u = 0$. Pick a neighborhood Y of x such that $Y \subset$ supp ϕ. Thus, $\phi \neq 0$ on Y. Observe that $\forall \psi \in C_0^\infty(Y)$, $\frac{\psi}{\phi} \in C_0^\infty(Y)$. Therefore we have

$$u(\phi) = u(\frac{\psi}{\phi}\phi) = (\phi u)(\frac{\psi}{\phi}) = 0 \qquad \forall \psi \in C_0^\infty(Y).$$

where the second identity comes from the definition of multiplication by smooth function in Definition 1.1.7, and the third identity comes from our assumption that $\phi u = 0$. Thus u vanishes on a neighborhood of x, which means by definition that $x \notin$ supp u.

In distribution theory, we are sometimes interested in distributions with compact support. Indeed we have a useful theorem for identification of such distributions.

Theorem 1.1.13.[4, Theorem 2.3.1] *The set of distributions in X with compact support is identical with the dual space of C^∞ with the topology defined by the semi-norms*

$$\|\phi\| = \sum_{|\alpha| \leq k} \sup_K |\partial^\alpha \phi|,$$

where K ranges over all compact subsets of X and k over all integers ≥ 0.

The identification can be even more concise for a distribution supported by a single point.

Theorem 1.1.14.[4, Theorem 2.3.4] *If u is a distribution of order k with support equal to $\{y\}$, then u has the form*

$$u(\psi) = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha \phi(y).$$

Closely related to the notion of support is the notion of singular support.

Definition 1.1.15. If $u \in \mathcal{D}'(X)$, then the singular suppport of u, denoted singsupp u, is the set of points in X having no open neighborhood to which the restriction of u is a C^∞ function.

Since every point in $X\backslash$singsupp u has a neighborhood where u is a C^∞ function, it follows from Theorem 1.1.10 that the restriction of u to $X\backslash$singsupp$u$ is a C^∞ function. This is not true for any larger open set than $X\backslash$singsupp u. Further we have an analogous characterization of the singular support to that of the support stated in Proposition 1.1.12.

Proposition 1.1.16. *$x \notin$ singsupp u if and only if there exists a nonnegative function $\phi \in C_0^\infty(X)$ such that $\phi(x) \neq 0$ and $\phi u \in C^\infty(X)$.*

Proof. Suppose that x is not in singsupp u. By definition, we can find a neighborhood Y of x to which the restriction of u is a C^∞ function. Consider a nonnegative function $\phi \in C_0^\infty(Y) \subset C_0^\infty(X)$ which is identically 1 around x. By construction we observe that $\phi(x) \neq 0$. We also note that the derivatives of ϕu exist and are continuous on Y for all orders as given in Definition 1.1.6. For instance if $\psi \in C_0^\infty(Y)$,

$$\partial_k(\phi u)(\psi) = -(\phi u)(\partial_k \psi) = -u(\phi \partial_k \psi)$$

and because $\phi\partial_k\psi \in C_0^\infty(Y)$ and u is smooth on Y, we see that $\partial_k(\phi u)$ exists and is continuous. Therefore $\phi u \in C^\infty(X)$.

Conversely, we assume that there exists a nonnegative function $\phi \in C_0^\infty(X)$ with $\phi(x) \neq 0$ and $\phi u \in C^\infty(X)$. Pick a neighborhood Y of x such that $Y \subset$ supp ϕ. Observe as $\phi \neq 0$ on Y that $\forall \psi \in C_0^\infty(Y)$, $\frac{\psi}{\phi} \in C_0^\infty(Y)$. Therefore we have

$$u(\psi) = u(\frac{\psi}{\phi}\phi) = (\phi u)(\frac{\psi}{\phi}) \qquad \forall \psi \in C_0^\infty(Y).$$

Because both ϕu and $\frac{\psi}{\phi}$ are C^∞ functions on Y, so must be u on Y. As u has a neighborhood of x to which its restriction is smooth, we conclude by definition that $x \notin$ singsupp u.

Sometimes we are interested in homogeneity or the degree of homogeneity of distributions.

Definition 1.1.17. A distribution u in $\mathbb{R}^n \setminus \{0\}$ is said to be homogeneous of degree a if it satisfies

$$\langle u, \phi \rangle = t^a \langle u, \phi_t \rangle \qquad \text{if } \phi \in C_0^\infty(\mathbb{R}^n \setminus 0), \qquad \phi_t(x) = t^n \phi(tx),\ t > 0 \tag{16}$$

If u is a distribution in $\mathbb{R}^n$ and (2.1.1) is valid for all $\phi \in C_0^\infty(\mathbb{R}^n)$ then u is said to be homogeneous of degree a in $\mathbb{R}^n$.

This definition of homogeneity of distributions is indeed a generalization of the homogeneity of continuous functions. To see this, we assume that $u \in C^(\mathbb{R}^n)$ is a homogeneous of degree a, i.e. $u(tx) = t^a u(x)$ for all $x \in \mathbb{R}^n \setminus 0$, $t > 0$. Note that u is identified as a distribution via (1.1.2), so we have

$$\langle u, \phi \rangle = \int u(x)\phi(x)\,\mathrm{d}x = \int u(tx)\phi(tx)t^n\,\mathrm{d}x = \int t^a u(x)\phi_t(x)\,\mathrm{d}x = t^a\langle u, \phi_t\rangle$$

It is clear that if ψ is a homogeneous C^∞ function in $\mathbb{R}^n \setminus 0$ of degree b then ψu is homogeneous of degree $a+b$. Since

$$\lambda \partial_j u = \partial_j \lambda u - \partial_j u = (a-1)\partial u$$

where $\lambda = \sum_{j=1}^{n} x_j \partial_j$, differentiation lowers the degree of homogeneity by one unit.

1.2 A review of the theory of differential manifolds

In the advanced theory of partial differential equations, we usually consider PDEs on various manifolds. Here we shall discuss basic facts on differential calculus on manifolds.

1.2.1 Vector bundles

Definition 1.2.1. A C^∞ real vector bundle over X with fiber dimension N is a C^∞ manifold V with

(i) a C^∞ map $\pi : V \to X$ called the projection,
(ii) a vector space structure in each fiber $V_x = \pi^{-1}(x)$,
(iii) local isomorphisms between V and the product of open subsets of X and $\mathbb{R}^N$.

Explicitly condition (iii) means that for each $x \in X$ there is an open neighborhood Y and a C^∞ map ψ of $V_Y = \pi^{-1}(Y)$ onto $Y \times \mathbb{R}^N$ such that ψ^{-1} is also in C^∞, $\psi(V_x) = x \times \mathbb{R}^N$ for every $x \in Y$, and the composed map

$$V_x \to x \times \mathbb{R}^N \to \mathbb{R}^N$$

is a linear isomorphism.

A vector bundle is thus a family of vector spaces V_x, $x \in X$, varying smoothly with x. Sometimes we call V_x the *fiber* of V at x. In case of the tangent bundle, the fibers are the tangent spaces.

Definition 1.2.2. The tangent space of X at x, denoted $T_x(X)$, is the space of real distribution densities t in X of order one with support at x and $t(1) = 0$.
The *tangent bundle* of X, denoted $T(X)$, is the subset of $X \times \mathbb{R}^{\dim X}$ defined by

$$T(X) = \{(x, v) \in X \times \mathbb{R}^{\dim X} : v \in T_x(X)\}$$

In order to understand the definition we first need to define distribution densities. Suppose that X_κ and $X_{\kappa'}$ are two coordinate patches equipped with the local coordinate systems $\kappa : X_\kappa \to \tilde{X}_\kappa$ and $\kappa' : X_{\kappa'} \to \tilde{X}_{\kappa'}$ respectively. If u is a continuous linear form on $C_0^\infty(X)$, it defines a distribution $u_\kappa \in \mathcal{D}'(\tilde{X}_\kappa)$ by

$$u_\kappa(\phi) = u(\phi \circ \kappa) \qquad \forall \phi \in C_0^\infty(\tilde{X}_\kappa). \tag{17}$$

Further if $\phi \in C_0^\infty(\kappa'(X_\kappa \bigcap X_{\kappa'}))$ then

$$u_{\kappa'}(\phi) = u(\phi \circ \kappa') = u(\phi \circ \psi_1 \circ \kappa) = u_\kappa(\phi \circ \psi_1)$$

where $\psi_1 = \kappa' \circ \kappa^{-1}$. This can be rewritten as

$$u_{\kappa'} = |\det \psi'| u_\kappa \circ \psi \tag{18}$$

in $\kappa'(X_\kappa \bigcap X_{kappa'})$ where $\psi = \kappa \circ \kappa'^{-1}$. Conversely we can show that given distributions $u_\kappa \in \mathcal{D}(\tilde{X}_\kappa)$ satisfying (18) we can find a continuous linear form u on $C_0^\infty(X)$ with (17). Therefore the continuous linear forms u on $C_0^\infty(X)$ can The system of distributions $u_\kappa \in \mathcal{D}(\tilde{X}_\kappa)$ satisfying (18). They are called *distribution densities.*

To justify the definition of the tangent space above we observe that if $X \subset \mathbb{R}^n$ then Theorem 1.1.11 gives for some $t_1, t_2, \cdots, t_n \in \mathbb{R}$

$$t(\phi) = \sum_1^n t_j \frac{\partial \phi(x)}{\partial x_j}$$

so $t(\phi)$ is the derivative of ϕ at x in the direction t, which it is natural to consider as a tangent vector at x. If X is a general manifold and x is in the coordinate patch X_κ equipped with the local cordinate systems $\kappa : X_\kappa \to \tilde{X}_\kappa$ then $t^\kappa(\phi) = t(\phi \circ \kappa)$ has the form

$$t^\kappa(\phi) = \sum t_j^\kappa \partial_j \phi(\kappa x), \qquad \phi \in C_0^\infty(\tilde{X}_\kappa).$$

Thus $T_x(X)$ is always a vector space of dimension $n = \dim X$, and we have identified $\bigcup_{x \in X_\kappa} T_x(X)$ with $\tilde{X}_\kappa \times \mathbb{R}^n$. If $X \in X_\kappa \cap X_{\kappa'}$, then

$$\langle t^{\kappa'}, \phi \rangle = \langle t^\kappa, \phi \circ f \rangle = \sum t_j^\kappa \partial_j (\phi \circ f)(\kappa x), \qquad \phi \in C_0^\infty(\kappa'(\tilde{X}_\kappa \cap \tilde{X}_{\kappa'}))$$

where $f = \kappa' \circ \kappa^{-1} : \kappa(\tilde{X}_\kappa \cap \tilde{X}_{\kappa'}) \to \kappa'(\tilde{X}_\kappa \cap \tilde{X}_{\kappa'})$. Thus

$$t_k^{\kappa'} = \sum \partial_j f_k t_j^\kappa, \qquad \text{that is,} \qquad t^{\kappa'} = f'(\kappa x) t^\kappa.$$

This is a C^∞ map so the atlas consisting of the maps

$$\bigcup_{x \in X_\kappa} T_x(X) \ni t \to (\kappa(x), t^\kappa) \in \tilde{X}_\kappa \times \mathbb{R}^n$$

makes $T(X)$ a C^∞ manifold.

We remark that the map from $T(X)|_{X_\kappa}$ to $\tilde{X}_\kappa \times \mathbb{R}^n$ above is a special case of a very general construction. If X_1 and X_2 are C^∞ manifolds and $f : X_1 \to X_2$ is a C^∞ map, then we have a map $f_* : T(X_1) \to T(X_2)$ with $f_* : T_x(X_1) \to T_{f(x)}(X_2)$ defined by

$$(f_* t)(\phi) = t(\phi \circ f), \qquad \phi \in C_0^\infty(X_2),\ t \in T(X_1). \tag{19}$$

Recall that we can associate a dual space for every vector field. This operation can be extended to vector bundles by performing the vector space operation *fiberwise.*

Definition 1.2.3. If V is a vector bundle over X, then its dual bundle V^* is defined by

$$V^* = \{(x, \phi) \in X \times \mathbb{R}^{\dim X} : \phi \in V_x^*(X)\}$$

where V_x is the fiber of V at $x \in X$ and $V_x^*(X)$ is its dual space.
In particular, the dual $T^*(X)$ of $T(X)$ is called the *cotangent bundle.*

For our analysis there are two important structures on $T^*(X)$. First, being a vector bundle, $T^*(X)$ is equipped with an $\mathbb{R}^+$-action: $\mathbb{R}^+ \times T^*(X) \ni (s, x, \xi) \to (x, s\xi)$. It is also a *symplectic manifold*, equipped with a *symplectic form* σ which we will define shortly.

If V is a vector bundle over X and $Y \subset X$ then a *section* u of V over Y is a map from Y to V with $\pi \circ u$ is the identity map.

For any $\phi \in C^k(X), k > 0$, the differential

$$T_x(X) \ni t \to t(\phi)$$

at x is an element $\mathrm{d}\phi(x) \in T_x^*(X)$, so ϕ defines a C^{k-1} section $\mathrm{d}\phi$ (or ϕ') of $T^*(X)$. An arbitrary section of $T^*(X)$ is called a *one form.*

We shall now look at $T^*(X)$ in local coordinates. If

$$X_\kappa \ni x \to \kappa(x) = (x_1, x_2, \cdots, x_n) \in \tilde{X}_\kappa \subset \mathbb{R}^n$$

is a local coordinate system, we have identified $T(X)|_{X_\kappa}$ with $\tilde{X}_\kappa \times \mathbb{R}^n$ so that (x, t) is the tangent vector

$$C_0^\infty(\tilde{X}_\kappa) \ni \phi \to (\sum t_j \frac{\partial}{\partial x_j})\phi(x).$$

Now $\langle t, \xi \rangle = \sum t_j \dfrac{\partial}{\partial x_j}(\sum \xi_j x_j)$, so in $T^*(X)$ the form $\sum \xi_j \mathrm{d}x_j$ at x corresponds to $(x, \xi), \xi \in \mathbb{R}^n$. If x_j and ξ_j are regarded as functions on $\tilde{X}_\kappa \times \mathbb{R}^n$ then $\sum \xi_j \mathrm{d}x_j$ can also be considered as a differential form there. If π is the projection $\tilde{X}_\kappa \times \mathbb{R}^n \to \tilde{X}_\kappa$ its value on the tangent vector t to $\tilde{X}_\kappa \times \mathbb{R}^n$ is $\langle \pi_* t, \xi \rangle$ where π_* is constructed as in (16).

The *canonical one form* ω on $T^*(X)$ is invariantly defined by letting its value on the tangent vector t at $\gamma \in T^*(X)$ be

$$\langle t, \omega \rangle = \langle \pi_* t, \gamma \rangle.$$

In the standard local coordinates in $T^*(X)$ we have $\omega = \sum \xi_j \mathrm{d}x_j$.

Let us discuss an alternative characterization of the canonical one form which usually turns out to be useful. If $f : X_1 \to X_2$ is a C^k map, we obtain a map

$$f^* : C^{k-1}(T^*(X_2)) \to C^{k-1}(T^*(X_1)),$$

called the *pullback* of the one form by f, if we define for a one form u in X_2 and a tangent vector $t \in T(X)$

$$< t, f^* u > = < f_* t, u > .$$

If $\phi \in C^1(X)$ then ϕ' is a map $X \to T^*(X)$ which we can use to pull back the canonical one form ω from $T^*(X)$ to X. As is obvious in the local coordinate representation of ω we obtain

$$(\phi')^* \omega = \mathrm{d}\phi. \tag{20}$$

The differential form

$$\sigma = \mathrm{d}\omega \tag{21}$$

is called the *symplectic form* of $T^*(X)$.

We close this section by introducing another important vector bundle called the *conormal bundle.*

Definition 1.2.4. If Y is a C^∞ submanifold of X then the conormal bundle $N^*(Y)$ of Y is defined as

$$N^*(Y) = \{\gamma \in T^*(X) : \pi_* \gamma = y \in Y \text{ and } \gamma \text{ vanishes on } T_y(Y)\}$$

If we introduce local coordinates $x = (x_1, x_2, \cdots, x_n)$ in X such that Y is defined by $x_1 = x_2 = \cdots = x_k = 0$ then $N^*(Y)$ is defined by $x_1 = x_2 = \cdots = x_k = \xi_{k+1} = \xi_{k+2} = \cdots = \xi_n = 0$ in the corresponding coordinates in $T^*(X)$. Thus $N(Y)$ is a vector bundle of total dimension n. $N^*(Y)$ is the dual of the bundle on Y with fiber $T_y(X)/T_y(Y)$ which is called the normal bundle, which is denoted $N(Y)$.

1.2.2 Differential operators

In order to make our analysis easier, we usually associate a function on a dual space to a differential operator and analyze the function instead of directly analyzing the operator.

Definition 1.2.5. Let $P = P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ be a differential operator with C^∞ coefficients of order m in an open set $X \subset \mathbb{R}^n$. The *characteristic polynomial* of P is defined by

$$P = P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha$$

The *principal symbol* P_m is defined by

$$P_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

The characteristic polynomial uniquely determines its corresponding differential operator. Its definition comes from the fact that a differential operator P acts on $\mathcal{S}(\mathbb{R}^n)$ by

$$P(x,D)u = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} P(x,\xi)u(y)\,\mathrm{d}y\,\mathrm{d}\xi. \tag{22}$$

Note that the principal symbol is homogeneous degree m. Its definition is in fact independent of the choice of local coordinates and bases in the bundle, for it means that if $\phi \in C^\infty(X)$ we have

$$P_m(x,\mathrm{d}\phi) = \lim_{t\to\infty} t^{-m}e^{-it\phi}Pe^{it\phi}.$$

and the right hand side is invariantly defined in the cotangent bundle. To prove the equivalence of (19) and (20) we just note that when (20) is evaluated in terms of local coordiates then we must let m derivatives fall on the exponential to get a non-zero contribution.

Now recall the definition of the symplectic form σ of $T^*(X)$ in (18). In the standard coordinates x,ξ in $T^*(X)$ we have

$$\sigma = \sum \mathrm{d}\xi_j \wedge \mathrm{d}x_j$$

which means that for two tangent vectors to $T^*(X)$ with the coordinates (t',τ') and (t'',τ'') the symplectic form is

$$\sum \begin{vmatrix} \tau'_j & t'_j \\ \tau''_j & t''_j \end{vmatrix} = \sum(\tau'_jt''_j - t'_j\tau''_j) = \langle t'',\tau'\rangle - \langle t',\tau''\rangle.$$

This is a non-degenerate bilinear form, that is, it vanishes for all (t'',τ'') if and only if $(t',\tau') = 0$. Since $\sigma = \mathrm{d}\omega$ we have of course $\mathrm{d}\sigma = 0$.

The symplectic form σ enables us to define a vector field on $T^*(X)$ that plays a fundamental role in our analysis.

Definition 1.2.6. If p is a real valued C^∞ function defined on $T^*(X)$, the *Hamilton vector field* H_p of p is defined by

$$\langle t, dp\rangle = \sigma(t,H_p), \qquad t \in T(T^*(X)).$$

In terms of local coordinates (20) means if $t = (t_x,t_\xi)$ and $H_p = (h_x,h_\xi)$ that

$$\langle t_x, \frac{\partial p}{\partial x}\rangle + \langle t_\xi, \frac{\partial p}{\partial \xi} >=< t_\xi, h_x > - < t_x, h_\xi\rangle,$$

that is, $h_x = \dfrac{\partial p}{\partial \xi}, h_\xi = -\dfrac{\partial p}{\partial x}$. Thus

$$H_p = \sum\left(\frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}\right) \tag{23}$$

in terms of local coordinates.

Having defined the Hamilton vector field, we can now introduce the notion of the ellipticity, the characteristic set and bicharacteristics.

Definition 1.2.7. A differential operator P in a manifold X of dimension n is said to be *elliptic* at $(\bar{x}, \bar{\xi}) \in T^*(X) \setminus o$ if its principal symbol p_m satisfies $p_m(\bar{x}, \bar{\xi}) = 0$ where o denotes the zero section of the cotangent bundle. The *characteristic set* $\Sigma(P)$ of P is the set of points at which P is not elliptic, that is, $\Sigma(P) = p_m^{-1}(\{0\})$. *Bicharacteristics* are integral curves of H_{p_m} inside Σ.

Note that $\Sigma(P)$ is locally a closed conic subset of $X \times \mathbb{R}^n$ in the sense that it is closed and $(\bar{x}, t\bar{\xi}) \in \Sigma(A)$ for every $t > 0$ whenever $(\bar{x}, \bar{\xi}) \in \Sigma(P)$.

2 Singularity propagation theorem in the boundaryless setting

2.1 Pseudodiffeerential operators

For the analysis of singularity propagation, we will work with a more general class of operators, called *pseudodifferential operators*, or ps.d.o.'s for short. Throughout this section all manifolds are assumed to be boundaryless.

2.1.1 Symbols

Recall that the principal symbol of a differential operator is a polynomial. We can similarly define the principal symbol of a pseudodifferential operator, but it belongs to the class of functions that generalize polynomials.

A polynomial p in ξ of degree at most m satisfies a bound

$$|p(\xi)| \leq C(1 + |\xi|)^m \qquad \forall \xi \in \mathbb{R}^n.$$

Since successive derivatives $D_\xi^\alpha p(\xi)$ are polynomials of degree $m - |\alpha|$, for any multiindex α, we get the family of estimates

$$|D_\xi^\alpha p(\xi)| \leq C(1 + |\xi|)^{m-|\alpha|} \qquad \forall \xi \in \mathbb{R}^n, \alpha \in \mathbb{N}_0^n.$$

If we consider the principal symbol $P_m(x, \xi)$ of a differential operator P with C^∞ coefficients the family of estimates above is replaced by

$$|D_x^\alpha D_\xi^\beta P(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \qquad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n, \alpha, \beta \in \mathbb{N}_0^n.$$

There is no particular reason to have the same number of x variables as of ξ variables, so in general we define:

Definition 2.1.1. The space $S^m(\mathbb{R}^p;\mathbb{R}^n)$ of symbols of order m (with coefficients in $C_0^\infty(\mathbb{R}^p)$ consists of those functions $a \in C^\infty(\mathbb{R}^p \times \mathbb{R}^n)$ satisfying all the estimates

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \text{ on } \mathbb{R}^p \times \mathbb{R}^n \qquad \forall \alpha \in \mathbb{N}_0^p,\ \beta \in \mathbb{N}_0^n. \tag{24}$$

We even define $S^m(\Omega;\mathbb{R}^n)$ when $\Omega \subset \mathbb{R}^p$ and $\Omega \subset cl(\text{int}(\Omega))$ as consisting of those $a \in C^\infty(\text{int}(\Omega) \times \mathbb{R}^n)$ satisfying (23) for all $(x,\xi) \in \text{int}(\Omega) \times \mathbb{R}^n$.

The estimate (24) can be rewritten

$$||a||_{N,m} = \sup_{(x,\xi)\in\text{int}(\Omega)\times\mathbb{R}^n} \max_{|\alpha|+|\beta|\leq N} (1+|\xi|)^{-m+|\beta|}|D_x^\alpha D_\xi^\beta a(x,\xi)| < \infty. \tag{25}$$

With these norms $S^m(\Omega;\mathbb{R}^n)$ is a Fréchet space.

Now we discuss some basic properties. First notice that there exists a constant C with

$$(1+|\xi|)^m \leq C(1+|\xi|)^{m'} \quad \forall \xi \in \mathbb{R}^n \iff m \leq m'.$$

Thus we have an inclusion

$$S^m(\Omega;\mathbb{R}^n) \subset S^{m'}(\Omega;\mathbb{R}^n) \qquad \forall m' \geq m. \tag{26}$$

Moreover this inclusion is continuous, since from (24), $||a||_{N,m'} \leq ||a||_{N,m}$ if $a \in S^m(\Omega;\mathbb{R}^n)$ and $m' \geq m$. Since these spaces increase with m we think of them as a *filtration* of the big space

$$S^\infty(\Omega;\mathbb{R}^n) = \bigcup_m S^m(\Omega;\mathbb{R}^n). \tag{27}$$

The *residual* space of this filteration is

$$S^{-\infty}(\Omega;\mathbb{R}^n) = \bigcap_m S^m(\Omega;\mathbb{R}^n).$$

We also note a very useful result. Namely

$$S^m(\Omega;\mathbb{R}^n) \cdot S^{m'}(\Omega;\mathbb{R}^n) \subset S^{m+m'}(\Omega;\mathbb{R}^n). \tag{28}$$

This can be proved directly using Leibniz' formula:

$$
\begin{aligned}
&\sup_{\xi}(1+|\xi|)^{-m-m'+|\beta|}\left|D_x^{\alpha}D_{\xi}^{\beta}(a(x,\xi)\cdot b(x,\xi))\right| \\
&\leq \sum_{\mu\leq\alpha,\gamma\leq\beta}\binom{\alpha}{\mu}\binom{\beta}{\gamma}\sup_{\xi}(1+|\xi|)^{-m+|\gamma|}\left|D_x^{\mu}D_{\xi}^{\gamma}a(x,\xi)\right| \times \sup_{\xi}(1+|\xi|)^{-m'+|\beta-\gamma|}\left|D_x^{\alpha-\mu}D_{\xi}^{\beta-\gamma}b(x,\xi)\right| \\
&< \infty
\end{aligned}
$$

The construction of the spaces of symbols as above is indeed a very general construction, and we can introduce many subspaces with additional conditions to the estimate (24). For the construction of such subspaces, we first introduce the notion of *asymptotic summability* of symbols.

Definition 2.1.2. Suppose $a_j \in S^{m-j}(\Omega;\mathbb{R}^n)$ for $j = 1, 2, \cdots$. The series a_j is said to be *asymptotically summable* if there exists $a \in S^m(\Omega;\mathbb{R}^n)$ such that

$$a - \sum_{j=0}^{N-1} a_j \in S^{m-N}(\Omega,\mathbb{R}^n) \qquad \forall N \in \mathbb{N}.$$

We write this relation as $a \sim \sum_{j=1}^{\infty} a_j$.

Indeed, we can show that any series of symbols with $a_j \in S^{m-j}(\Omega;\mathbb{R}^n)$ for $j = 1, 2, \cdots$ is asymptotically summable and the asymptotic sum is well-defined up to an additive term in $S^{-\infty}(\Omega,\mathbb{R}^n)$.

Now we introduce one important class of subspaces of $S^m(\Omega,\mathbb{R}^n)$.

Definition 2.1.3. For any $m \in \mathbb{C}$, the space of *polyhomogeneous symbols*, denoted by $S^m_{\text{ph}}(\Omega,\mathbb{R}^n)$, is defined by the requirement that $a \in S^m_{\text{ph}}(\Omega,\mathbb{R}^n)$ if and only if there exist elements $a_{m-j} \in S^{\mathcal{R}e\,m}(\Omega;\mathbb{R}^n)$ which are homogeneous of degree $m-j$ on the set $\{(x,\xi) \in \Omega \times \mathbb{R}^n : |\xi| \geq 1\}$ for all $j \in \mathbb{N}_0$ such that $a \sim \sum_j a_{m-j}$.

Clearly we have

$$S^m_{\text{ph}}(\Omega,\mathbb{R}^n) \cdot S^{m'}_{\text{ph}}(\Omega,\mathbb{R}^n) \subset S^{m+m'}_{\text{ph}}(\Omega,\mathbb{R}^n)$$

because the asymptotic expansion of the product is given by the formal product of the asymptotic expansion. In fact we have equality here, because

$$b(x,\xi) = (1+|\xi|^2)^{\frac{m}{2}} \in S^m_{\text{ph}}(\Omega,\mathbb{R}^n)$$

and multiplication by b is an isomorphism of the space $S^0_{\text{ph}}(\Omega,\mathbb{R}^n)$ onto $S^m_{\text{ph}}(\Omega,\mathbb{R}^n)$.

2.1.2 Pseudodifferential operators

Recall from (22) that the action of a differential a differential operator with C^∞ coefficients on $\mathcal{S}(\mathbb{R}^n)$ can be written

$$P(x,D)u = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} P(x,\xi)u(y)\mathrm{d}y\mathrm{d}\xi$$

We now consider an analogous integral

$$A(x,D)u = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} a(x,y,\xi)u(y)\mathrm{d}y\mathrm{d}\xi, \qquad u \in \mathcal{S}(\mathbb{R}^n). \tag{29}$$

where we assume that, for some $w, m \in \mathbb{R}$,

$$a(x,y,\xi) = (1+|x-y|^2)^{\frac{w}{2}}\tilde{a}(x,y,\xi) \qquad \tilde{a} \in S^m(\mathbb{R}^{2n};\mathbb{R}^n).$$

We can show that if $a \in C^\infty(\mathbb{R}^{2n} \times \mathbb{R}^n)$ then $a \in (1+|x-y|^2)^{\frac{w}{2}} S^m(\mathbb{R}^{2n};\mathbb{R}^n)$ if and only if

$$|D_x^\alpha D_y^\beta D_\xi^\gamma a(x,y,\xi)| \le C_{\alpha,\beta,\gamma}(1+|x-y|)^w(1+|\xi|)^{m-|\gamma|} \qquad \forall \alpha,\beta,\gamma \in \mathbb{N}_0^n.$$

If $m < -n$ then, for each $u \in \mathcal{S}(\mathbb{R}^n)$, the integral in (29) is absolutely convergent, locally uniformly in x, since

$$\begin{aligned} |a(x,y,\xi)u(y)| &\le C(1+|x-y|)^w(1+|\xi|)^m(1+|y|)^{-N} \\ &\le C(1+|x|)^w(1+|\xi|)^m(1+|y|)^m. \end{aligned}$$

Here we used the following simple consequence of the triangle inequality

$$(1+|x-y|) \le (1+|x|)(1+|y|)$$

from which it follows that

$$(1+|x-y|)^w \le \begin{cases} (1+|x|)^w(1+|y|)^w & \text{if } w > 0 \\ (1+|x|)^w(1+|y|)^{-w} & \text{if } w \le 0 \end{cases}$$

Thus, we conclude that, provided $m < -n$,

$$A: \mathcal{S}(\mathbb{R}^n) \to (1+|x|^2)^{\frac{w}{2}} C^0(\mathbb{R}^n).$$

To show that, for general m, A exists as an operator, we need the easy part of thefollowing important theorem.

Theorem 2.1.4. (Schwartz kernel theorem) *If X_1 and X_2 are open sets in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, every $K \in \mathcal{D}'(X_1 \times X_2)$ defines a linear map $\mathcal{K}$ from $C_0^\infty(X_2)$ to $\mathcal{D}'(X_1)$ by*

$$< \mathcal{K}\phi, \psi >= K(\psi \otimes \phi), \qquad \psi \in C_0^\infty(X_1), \phi \in C_0^\infty(X_2). \tag{30}$$

.

The map is continuous in the sense that $\mathcal{K}\phi_j \to 0$ in $\mathcal{D}'(X_1)$ if $\phi_j \to 0$ in $C_0^\infty(X_2)$. Conversely, to every such linear map $\mathcal{K}$ there is one and only one distribution K such that (30) *is valid.*

Here $\otimes$ denotes the tensor product, that is, $\psi \otimes \phi$ is a map on $X_1 \times X_2$ defined by $\psi \otimes \phi(x_1, x_2) = \psi(x_1)\phi(x_2)$. K is called the *kernel* of $\mathcal{K}$.

Now we construct the Schwartz kernel of A.

Lemma 2.1.5. *The map* $I : (1+|x-y|^2)^{\frac{w}{2}} S^m(\mathbb{R}^{2n}; \mathbb{R}^n) \to (1+|x|^2+|y|^2)^{\frac{w}{2}} C^0(\mathbb{R}^{2n})$ *defined for* $m < -n$ *as a convergent integral*

$$I(a) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x, y, \xi)\, \mathrm{d}\xi \tag{31}$$

extends by continuity to

$$I : \ (1 + |x - y|^2)^{\frac{w}{2}} S^m(\mathbb{R}^{2n}; \mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^{2n}) \tag{32}$$

for each $w, m \in \mathbb{R}$ *in the topology of* $S^{m'}(\mathbb{R}^{2n}; \mathbb{R}^n)$ *for any* $m' > m$.

Proof. Since we already have the density of $S^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ in $S^m(\mathbb{R}^{2n}; \mathbb{R}^n)$ in the topology of $S^{m'}(\mathbb{R}^{2n}; \mathbb{R}^n)$ for any $m' > m$, we only need to show the continuity of the map (31) on this residual subspace with respect to the topology of $S^{m'}(\mathbb{R}^{2n}; \mathbb{R}^n)$ for any m' which we may as well write as m. What we shall show is that, for each $w, m \in \mathbb{R}$, there are integers $N, k \in \mathbb{N}$ such that, in terms of the norms in (25) and (13)

$$|I(a)(\phi)| \leq C \|\tilde{a}\|_{N,m} \|\phi\|_k \tag{33}$$

whenever $\phi \in \mathcal{S}^{2n}$, $a \in (1 + |x - y|^2)^{\frac{w}{2}} \tilde{a}$, $\tilde{a} \in S^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$.

To see this we just use integration by parts. Set $\tilde{\phi}(x, y) = (1 + |x - y|^2)^{\frac{w}{2}} \phi(x, y)$. Observe that

$$\begin{aligned} (1 + \xi \cdot D_x) e^{i(x-y)\cdot\xi} &= (1 + |\xi|^2) e^{i(x-y)\cdot\xi} \\ (1 - \xi \cdot D_y) e^{i(x-y)\cdot\xi} &= (1 + |\xi|^2) e^{i(x-y)\cdot\xi}. \end{aligned}$$

Thus we can write for any $q \in \mathbb{N}$,

$$\begin{aligned} I(a)(\phi) &= \iint (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} (1 + |\xi|^2)^{-2q} (1 - \xi \cdot D_x)^q (1 + \xi \cdot D_y)^q [\tilde{a}(x, y, \xi) \tilde{\phi}(x, y)]\, \mathrm{d}\xi\, \mathrm{d}x\, \mathrm{d}y \\ &= \sum_{|\gamma| \leq 2q} \iint \Big(\int e^{i(x-y)\cdot\xi} a_\gamma^{(q)}(x, y, \xi)\, \mathrm{d}\xi \Big) D^\gamma_{(x,y)} \tilde{\phi}(x, y)\, \mathrm{d}x\, \mathrm{d}y. \end{aligned}$$

Here the $a_\gamma^{(q)}$ arise by expanding the powers of the operator

$$(1 - \xi \cdot D_x)^q (1 + \xi \cdot D_y)^q = \sum_{|\mu|, |\nu| \leq q} C_{\mu,\nu} \xi^{\mu+\nu} D_x^\mu D_y^\nu.$$

Thus $a_\gamma^{(q)}$ arises from terms in which $2q - |\gamma|$ derivatives act on $\tilde{a}$ so it is of the form

$$\begin{aligned} a_\gamma &= (1 + |\xi|^2)^{-2q} \sum_{|\mu| \leq |\gamma|, |\gamma| \leq 2q} C_{\mu,\gamma} \xi^\gamma D^\mu_{(x,y)} \tilde{a} \\ \Longrightarrow \|a_\gamma\|_{N,m} &\leq C_{m,q,N} \|\tilde{a}\|_{N+2q, m+2q} \qquad \forall m, N, q. \end{aligned}$$

So if we take q so that we have $-2q+m < -n$ and use the integrability of $(1+|x|+|y|)^{-2n-1}$ on $\mathbb{R}^{2n}$, then

$$|I(a)(\phi)| \leq C\|\tilde{a}\|_{2q,m}\|\tilde{\phi}\|_{2q+2n+1} \leq C\|\tilde{a}\|_{2q,m}\|\phi\|_{2q+w+2n+1}.$$

This is the estimate (33), which proves the desired continuity.

With its Schwartz kernel constructed, we can finally show that A exists as an operator.

Theorem 2.1.6. *If $a \in (1+|x-y|^2)^{\frac{w}{2}}S^m(\mathbb{R}^{2n};\mathbb{R}^n)$ then the operator A, with Schwartz kernel $I(a)$ as defined in Lemma 2.1.5, is a continuous linear map*

$$A:\ \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n).$$

We shall denote by $\Psi^m(\mathbb{R}^n)$ the linear space of operators corresponding to $a \in (1+|x-y|^2)^{\frac{w}{2}}S^m(\mathbb{R}^{2n};\mathbb{R}^n)$. We call these operators *pseudodifferential operators.*

To transfer the definition of pseudodifferential operators to manifolds we need invariance under diffeomorphisms between open subsets. Once we have local coordinate invariance, we can define the spaces $\Psi^m(X)$ for any manifold $X \subset \mathbb{R}^n$.

Proposition 2.1.7. *Suppose that $F:\ \Omega \to \Omega'$ is a diffeomorphism where Ω and Ω' are two open subsets of $\mathbb{R}^n$. Let $\Psi^m_c(\Omega')$ be the set of all $A \in \Psi^m(\mathbb{R}^n)$ which has kernel satisfying that supp A is contained in $\Omega \times \Omega'$. Define $A_F u = F^*A(F^{-1})^*(u|_\Omega)$ for $A \in \Psi^m_c(\Omega')$. Then $A_F \in \Psi^m_c(\Omega)$ and the map $\Psi^m_c(\Omega') \ni A :\to A_F \in \Psi^m_c(\Omega)$ is an isomorphism.*

Now we define the principal symbol of a pseudodifferential operator.

Consider the Schwartz kernel I of a pseudodifferential opertator A as in Lemma 2.1.5. It can be shown that the special case $w=0$ and $\partial_y a = 0$ gives an isomorphism

$$q_L:\ S^m(\mathbb{R}^n;\mathbb{R}^n) \to \Psi^m(\mathbb{R}^n). \tag{34}$$

The map q_L is called the *left quantization map* and its inverse σ_L is called the *left full symbol map.* We can indeed give an explicit formula for the left quantization.

$$q_L(a) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,\xi)\,\mathrm{d}\xi, \qquad a \in S^m(\mathbb{R}^n;\mathbb{R}^n). \tag{35}$$

As well as the left quantization map leading to the isomorphism (34) there is a *right quantization map*

$$q_R(a) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(y,\xi)\,\mathrm{d}\xi, \qquad a \in S^m(\mathbb{R}^n;\mathbb{R}^n). \tag{36}$$

which also gives an isomorphism between $S^m(\mathbb{R}^n;\mathbb{R}^n)$ and $\Psi^m(\mathbb{R}^n)$. Its inverse σ_R is called the *right full symbol map.*

For a pseudodifferential operator $A \in \Psi^m(\mathbb{R}^n)$, we can relate $\sigma_L(A)$ and $\sigma_R(A)$ using the full asymptotic expansion.

$$\sigma_R(A)(x,\xi) \sim \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha \sigma_L(A)(x,\xi) \tag{37}$$

Notice that the term that correspond to α is in $S^{m-|\alpha|}(\mathbb{R}^n;\mathbb{R}^n)$. From this relation therefore we deduce that $\sigma_L(A) - \sigma_R(A) \in S^{m-1}(\mathbb{R}^n;\mathbb{R}^n)$. For $a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$, we write $[a]$ for its equivalence class in the quotient space $S^m(\mathbb{R}^n;\mathbb{R}^n)/S^{m-1}(\mathbb{R}^n;\mathbb{R}^n)$.

Definition 2.1.8. The *principal symbol map* $\sigma_m : \Psi^m(\mathbb{R}^n) \to S^m(\mathbb{R}^n;\mathbb{R}^n)/S^{m-1}(\mathbb{R}^n;\mathbb{R}^n)$ is defined by

$$\sigma_m(A) = [\sigma_L(A)] = [\sigma_R(A)]$$

The principal symbol can be defined as above for any $A \in \Psi^m(X)$ where X is a manifold. The principal symbol σ_m is homogeneous degree m on $T^*(X) \setminus o$ where o denotes the zero section of the cotangent bundle. It is indeed easy to verify that this definition extends the definition of the principal symbol for differential operators in Definition 1.2.5.

Below we list some basic algebraic properties of the space $\Psi^m(X)$.

Theorem 2.1.9. *Suppose X is a manifold of dimension n.*

(i) *The spaces $\Psi^m(X)$ increases with m. That is, $\Psi^m(X) \subset \Psi^{m'}(X)$ whenever $m \le m'$.*

(ii) *If $A \in \Psi^m(X)$ then $A^* \in \Psi^m(X)$.*

(iii) *$\Psi^m(X) \cdot \Psi^{m'}(X) \subset \Psi^{m+m'}(X)$. That is, if $A \in \Psi^{m'}(X)$, $B \in \Psi^{m'}(X)$ then $AB \in \Psi^{m+m'}(X)$.*

Here A^* denotes the adjoint of A as a linear operator. The adjoint is taken with respect to any inner product given by a smooth density. The property (iii) is sometimes refered as the composition property of pseudodifferential operators. The properties (ii) and (iii) can be also stated in a single statement.

(iii)′ $\Psi^\infty(X) = \bigcup_m \Psi^m(X)$ is an order-filtered $*$-algebra.

Observe that these properties are analogous to the properties of the symbol spaces we discussed above. Indeed (i) and (iii) are consequences of (26) and (28) respectively.

The space of pseudodifferential operators also exihibits L^2 boundedness.

Theorem 2.1.10. *If $A \in \Psi^0(X)$ where X is a compact manifold, by continuity from $\mathcal{S}(X)$, A defines a bounded linear operator*

$$A : L^2(X) \to L^2(X).$$

We say an element of $\Psi^m(X)$ is *elliptic* if it is invertible modulo an error in $\Psi^{-\infty}(X)$ with the approximate inverse of order $-m$. In other words, $A \in \Psi^m(X)$ is elliptic if and only if there exists $B \in \Psi^{-m}(X)$ such that $AB - Id \in \Psi^{-\infty}(X)$ where Id is the identity operator. We call B the *left parametrix* of A. It can be shown that if B is the left parametrix of A, it also satisfies that $BA - Id \in \Psi^{-\infty}(X)$. Thus we say that the left parametrix of A is also the right parametrix of A, and hence we simply say that B is the *parametrix* of A. Hence the parametrix can be thought as the approximate inverse of a pseudodifferential operator. We also remark that the ellipticity of an operator is equivalent to the invertibility of its principal symbol.

For elliptic operators, we can also find their approximate square root as suggested in the following proposition.

Proposition 2.1.11. *Suppose $A \in \Psi^m(X)$ with $m > 0$ is elliptic and self-adjoint, i.e. $A = A^*$, with a positive principal symbol. Then we can find $B \in \Psi^{\frac{m}{2}}(X)$ with $B = B^*$ such that*

$$A = B^2 + G, \qquad G \in \Psi^{-\infty}(X).$$

So far we have been considering operators $A \in \Psi^m(X)$ which correspond, via (29), to amplitudes satisfying the symbol estimates (24). As already remarked, there are many varinats of these estimates and corresponding spaces of symbols, namely the spaces of one-step polyhomogeneous symbols. Just as we constructed the space of pseudodifferential operators via symbols, we can also construct a family of subspaces of pseudodifferential operators via polyhomogeneous symbols.

Definition 2.1.12. The spaces of *classical pseudodifferential operators* on $\mathbb{R}^n$, denoted by $\Psi^m_{\text{ph}}(\mathbb{R}^n)$, are defined by the condition that the kernel of $A \in \Psi^m_{\text{ph}}(X)$ should be of the form $I(a)$ as in Theorem 2.1.5 for some $a \in (1 + |x - y|^2)^{\frac{w}{2}} S^m_{\text{ph}}(\mathbb{R}^{2n}; \mathbb{R}^n)$.

Note that if m is real, $\Psi^m_{\text{ph}}(\mathbb{R}^n)$ is a subspace of $\Psi^m(\mathbb{R}^n)$. Therefore the definition can be extended for manifolds X as subspaces of the spaces $\Psi^m(X)$ and we can even define their principal symbols. We can also show that Theorem 2.1.9 holds for the space of classical pseudodifferential operators. Sometimes we use the notation $\Psi^m(\mathbb{R}^n)$ instead of $\Psi^m_{\text{ph}}(\mathbb{R}^n)$ if there is no confusion.

For elements of $A \in \Psi^m_{\text{ph}}(X)$, the principal symbol $\sigma_m(A)$ has a preferred class of representatives, namely the leading term in the expansion of $\sigma_L(A)$. In other words, we often write

$$\sigma_m(A) = \rho(\xi) a_m(x, \xi) \qquad \text{mod } S^{m-1}_{\text{ph}}(\mathbb{R}^n; \mathbb{R}^n)$$

where $\rho(\xi) = 1$ if $|\xi| \geq 1$ and $\rho(\xi) = 0$ if $|\xi| \leq \frac{1}{2}$. It is even natural to identify the principal symbol with $a_m(x,\xi)$ as a *homogeneous* function.

2.2 Wavefront set

Now we will develop a notion of singularity of a function or distribution. One of the most commonly used notion is the *wavefront set*, which locates at which points and in which direction a given function is not smooth.

2.2.1 Characteristic set

Recall from Definition 1.2.6 that the characteristic set of a differential operator is the set of points at which the operator is not elliptic. In order to define the characteristic set of a pseudodifferential operator in the same way, we need to generalize the notion of ellipticity.

Definition 2.2.1. If $A \in \Psi^m(X)$ with the principal symbol σ_m, A is said to be *elliptic* at $(\bar{x},\bar{\xi}) \in T^*(X) \setminus o$ if there exists $\epsilon > 0$ such that

$$|\sigma_m(A)(x,\xi)| \geq \epsilon|\xi|^m \text{ whenever } |x-\bar{x}| \leq \epsilon,\ |\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\xi|}| \leq \epsilon,\ |\xi| \geq \frac{1}{\epsilon}.$$

The *characteristic set* $\Sigma(A)$ of A is defined as for differential operators by

$$\Sigma(A) = \{(\bar{x},\bar{\xi}) \in T^*(X) \setminus o :\ A \text{ is not elliptic at } (\bar{x},\bar{\xi})\}$$

Bicharacteristics are integral curves of $H_{\sigma_m(A)}$ inside $\Sigma(A)$.

We can easily verify that $\Sigma(A)$ is a closed conic subset of $X \times (\mathbb{R}^n \setminus \{0\})$. In other words, it is closed and $(\bar{x},\bar{\xi}) \in \Sigma(A)$ implies $(\bar{x},t\bar{\xi}) \in \Sigma(A)$ for every $t > 0$.

The characteristic set of a pseudodifferential operator has an additional property. Since the product of two symbols is only elliptic at $(\bar{x},\bar{\xi})$ if they are both elliptic there, it follows from the composition property of pseudodifferential operators that

$$\Sigma(AB) = \Sigma(A) \cup \Sigma(B) \qquad A \in \Psi^m(X), B \in \Psi^{m'}(X) \tag{38}$$

Observe that the definition of bicharacteristics in Definition 1.2.7 makes sense as well for pseudodifferential operators as for differential operators because the pricipal symbol of a pseudodifferential operator can be regarded as a map on the cotangent bundle and we can consider its Hamilton vector field.

The Hamilton vector field turns out to be extremely useful for our commutator analysis of pseudodifferential operators.

Theorem 2.2.2.(See [1], page 4) *Suppose $A \in \Psi^m(X)$ and $B \in \Psi^{m'}(X)$ are pseudodifferential operators where X is a manifold. Then $[A,B] = AB - BA \in \Psi^{m+m'-1}(X)$ and*

$$\sigma_{m+m'-1}(i[A,B]) = H_{\sigma_m(A)}\sigma_{m'}(B)$$

2.2.2 Wavefront set

Now we are ready to define the wavefront set of a distribution. Among several different versions, we adapt the following definition.

Definition 2.2.3. If $u \in C_0^{-\infty}(X) = \{u \in \mathcal{S}'(X) : \text{supp}u \text{ is a compact subset of } X\}$ then the *wavefront set* $WF(u)$ is defined by

$$WF(u) = \bigcap_{A \in \Psi^0(X), Au \in C^\infty(X)} \Sigma(A).$$

The definition can be extended to general distributions by setting

$$WF(u) = \bigcup_{\phi \in C_0^\infty} WF(\phi u).$$

For $u \in C_0^{-\infty}(X)$, the wavefront set $WF(u)$ is always a closed conic set in $X \times \mathbb{R}^n \setminus \{0\}$, being the intersection of such sets. For $u \in \mathcal{D}'(X)$, $WF(u)$ is a closed conic set in $X \times \mathbb{R}^n \setminus \{0\}$ as well, being the union of such sets.

The definition can be further extended to general distributions. Even for general distributions, the wave front set is still a closed conic set in $X \times \mathbb{R}^n \setminus \{0\}$.

The following theorem gives an equivalent definition of the wavefront set.

Theorem 2.2.4. *Suppose that $u \in \mathcal{D}'(X)$. $(x, \xi) \in T^*(X) \setminus o$ is not in $WF(u)$ if and only if there exists $A \in \Psi^0(X)$ such that A is elliptic at (x, ξ) and $Au \in C^\infty(X)$.*

We will find this definition more useful when we define the wavefront set of a pseudodifferential operator on a manifold with boundary because it is simpler.

One of the most important facts about the wavefront set is that it is a refinement of singsupp u.

Theorem 2.2.5. *Let $\pi : X \times \mathbb{R}^n \setminus \{0\} \ni (x, \xi) \to x \in X$ be the projection in X. Then we have*

$$\pi(WF(u)) = \text{singsupp } u \qquad \forall u \in \mathcal{D}'(X).$$

Therefore the wavefront set contains all information in the singular support. Recall that $X \setminus \text{singsupp } u$ is the largest open set to which the restriction of $u \in \mathcal{D}'(X)$ is a C^∞ function. One immediate but important consequence of the theorem is that $WF(u)$ is empty if and only if u is a C^∞ function.

Next we shall consider the notion of the wave front set of a pseudodifferential operator.

Definition 2.2.6. If $a \in S^m(\Omega; \mathbb{R}^n)$ we define the *cone support* of u by

$$\begin{aligned}(\text{conesupp } a)^{\text{c}} &= \{(\bar{x}, \bar{\xi}) \in \Omega \times (\mathbb{R}^n \setminus \{0\}) : \exists \epsilon > 0 \text{ s.t. } \forall M \in \mathbb{R}, \exists C_M \text{ with} \\ &\quad |a(x,\xi)| \le C_M \langle \xi \rangle^{-M} \text{ if } |x - \bar{x}| \le \epsilon, |\frac{\xi}{|\xi} - \frac{\bar{\xi}}{|\bar{\xi}|}| \le \epsilon\}\end{aligned}$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$.

This is clearly a closed conic subset in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}$. By definition symbol decays rapidly outside this cone, in fact even more is true.

Proposition 2.2.7. *Suppose that $a \in S^\infty(\Omega; \mathbb{R}^n)$. If $(\bar{x}, \bar{\xi}) \notin$ conesupp a, we can find $\epsilon > 0$ such that $\forall M \in \mathbb{R}$, $\exists C_M$ with*

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_M \langle \xi \rangle^{-M} \text{ if } |x - \bar{x}| < \epsilon, |\frac{\xi}{|\xi} - \frac{\bar{\xi}}{|\bar{\xi}|}| < \epsilon$$

For a differential operator $A \in \Psi^m(X)$ we associated two symbols of order m, namely $\sigma_L(A)$ and $\sigma_R(A)$. From their asymptotic relation (37) we can show that their cone supports coincide. Because the principal symbol $\sigma_m(A)$ is defined as a representative of the equivalence class of $\sigma_L(A)$ and $\sigma_R(A)$, we can define the *operator wave front set* as follows.

Definition 2.2.8. For every $A \in \Psi^m(X)$, the *wavefront set $WF'(A)$* of A is defined by

$$WF'(A) = \text{conesupp } \sigma_L(A) = \text{conesupp } \sigma_R(A).$$

Sometimes we use a terminology *essential support* instead of wavefront set.

In our analysis we usually need to consider the wavefront set of a sum or a composition of operators and the following theorem turns out to be very useful.

Theorem 2.2.9. *Suppose X is a manifold and $A \in \Psi^m(X)$ and $B \in \Psi^{m'}(X)$. We have following inclusions.*

$$\begin{aligned}WF'(A \circ B) \subset WF'(A) \cap WF'(B) \\ WF'(A + B) \subset WF'(A) \cup WF'(B)\end{aligned}$$

Now we can consider the relationship between these two notions of wavefront set.

Theorem 2.2.10. *Pseudodifferential operators are microlocal in the sense that*

$$WF(Au) \subset WF'(A) \cap WF(u) \qquad \forall A \in \Psi^\infty(X),\ u \in \mathcal{D}'(X)$$

Here we notice in particular that we have the following inclusion.

$$WF(Au) \subset WF(u) \qquad \forall A \in \Psi^{\infty}(X),\ u \in \mathcal{D}'(X) \tag{39}$$

Indeed we can show a weak converse of (39).

Theorem 2.2.11. *For any $u \in \mathcal{D}'(X)$ and any $A \in \Psi^m(X)$*

$$WF(u) \subset WF(Au) \cup \Sigma(A).$$

For the proof, we first refine the notion of a parametrix for an elliptic operator to that of a *microlocal parametrix.*

Lemma 2.2.12. *If $A \in \Psi^m(X)$ and $z \notin \Sigma(A)$ then there exists $B \in \Psi^{-m}(X)$ such that*

$$z \notin WF'(Id - AB), \qquad z \notin WF'(Id - BA)$$

where Id denotes the identity operator. B is called the microlocal parametrix at z.

Proof. We write $z = (\bar{x}, \bar{\xi})$ where $\bar{x} \in X$ and $\bar{\xi}$ is its dual variable. We can construct a symbol $\gamma_\epsilon \in S^0(X, \mathbb{R}^n)$ with its support in

$$\bar{C} = \{(x,\xi) \in T^*(X) :\ |x - \bar{x}| \le \epsilon,\ |\xi| \ge \frac{1}{2\epsilon},\ |\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\bar{\xi}|}| \le \epsilon\} \tag{40}$$

and is identically equal to one, and hence elliptic, on a smaller set

$$\tilde{C} = \{(x,\xi) \in T^*(X) :\ |x - \bar{x}| \le \frac{\epsilon}{2},\ |\xi| \ge \frac{1}{\epsilon},\ |\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\bar{\xi}|}| \le \frac{\epsilon}{2}\}.$$

Now we define L_ϵ by $\sigma_L(L_\epsilon) = \gamma_\epsilon$. Then for any $\epsilon > 0$,

$$z \notin WF'(Id - L_\epsilon),\ WF'(L_\epsilon) \subset \{(x,\xi) \in T^*(X) :\ |x - \bar{x}| \le \epsilon,\ |\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\bar{\xi}|}| \le \epsilon\}.$$

Pick $G_{2m} \in \Psi^{2m}(X)$ so that it is a globally elliptic operator with positive principal symbol. Namely, we can simply take $\sigma_L(G_{2m}) = (1 + |\xi|^2)^m$. Now consider the operator $J = (Id - L_\epsilon)G_{2m} + A^*A \in \Psi^{2m}(X)$. The principal symbol of J is $(1 - \gamma_\epsilon)(1 + |\xi|^2)^m$ which is globally elliptic if $\epsilon > 0$ is small enough, so we can find a parametrix $H \in \Psi^{-2m}(X)$ of J. Now if we take $B = HA^* \in \Sigma^{-m}(X)$, we have

$$BA - Id = HA^*A - Id = (HJ - ID) + H(Id - L_\epsilon)G_{2m}A.$$

The first term on the right is in $\Psi^{-\infty}$ while z is not in $WF'(Id - L_\epsilon)$ and hence not in the operator wavefront set of the second term. Therefore z is not in $WF'(Id - BA)$. Similarly we can also show that z is not in $WF'(Id - AB)$, which completes the proof of the lemma.

Theorem 2.2.10 is then easily derived from Lemma 2.2.12.

Proof of Theorem 2.2.11. If $(\bar{x}, \bar{\xi}) \in T^*(X)$ is not in $\Sigma(A)$ then, by definition, A is elliptic at $(\bar{x}, \bar{\xi})$. Thus, by Lemma 2.2.11, A has a microlocal parametrix B, so

$$u = BAu + Su,\ (\bar{x}, \bar{\xi}) \notin WF'(S).$$

It follows that $(\bar{x}, \bar{\xi}) \notin WF(Au)$ implies that $(\bar{x}, \bar{\xi}) \notin WF(u)$ proving the theorem.

Below we list some important immediate consequences of Theorem 2.2.11.

Corollary 2.2.13. *Suppose that* $Au \in C^\infty(X)$ *where* $u \in \mathbb{D}'(X)$ *and any* $A \in \Psi^m(X)$. *Then* $WF(u) \subset \Sigma(A)$.

Corollary 2.2.14. *If* $A \in \Psi^m(X)$ *is (globally) elliptic, that is, A is elliptic at all points in* $T^*(X) \setminus o$ *where o is the zero section of the cotangent bundle, then*

$$WF(u) = WF(Au), \qquad u \in \mathbb{D}'(X).$$

In particular we have singsupp u = singsupp Au, $\qquad u \in \mathbb{D}'(X)$.

Corollary 2.2.13 immediately follows from Theorem 2.2.11 if we recall that the wavefront set of a distribution is empty if and only if the distribution is a C^∞ function. If $A \in \Psi^m(X)$ is globally elliptic, its characteristic set must be empty by definition, so the first identity of Corollary 2.2.14 follows from Theorem 2.2.11. Then we apply Theorem 2.2.5 to deduce the second identity of Corollary 2.2.14.

2.3 Propagation of singularities

Here we state the main facts about the analysis of pseudodifferential operators on manifolds without boundary.

Theorem 2.3.1. (See [1], page 5) *Suppose* $A \in \Psi^m(X)$, $a = \sigma_m(A)$ *is real,* $u \in \mathcal{D}'(X)$ *and* $Au = 0$.

(i) $WF(u) \subset \Sigma(A)$

(ii) $WF(u)$ *is a union of maximally extended bicharacteristics in* $\Sigma(A)$. *That is, if* $q \in WF(u)$ *then so is the whole bicharacteristic through* q.

Observe that (i) is just a restatement of Corollary 2.2.13. It is restated here for analogy with the manifold with boundary or corners.

We also note that (ii) may be vacuous; indeed, if H_a is *radial*, i.e. tangent to the orbits of the $\mathbb{R}^+$-action, then it does not give any information on $WF(u)$, as the latter is already known to be conic. Such points are called radial points, and in recent work, they have been extensively analyzed under nondegeneracy assumptions. If A is the wave operator, there are no radial points in $\Sigma(A)$, but such points are very important in scattering theory.

3 Singularity propagation theorem of wave equation in the manifold with boundary

3.1 b-calculus

For pseudodifferential operators on a manifold with corners or boundaries, the definitions of the wavefront set and the bicharacteristics change significantly. These modifications require us to extend various concepts of differential calculus on manifolds. Because developing these concepts is just a repetition of what we have done to define the set of pseudodifferential operators in the boundaryless setting with more complexity, we will mostly focus on describing their main properties and studing how these properties are analogous to those in the boundaryless setting rather than stating concise definitions.

Our goal is to construct an algebra of pseudodifferential operators on a manifold X with boundary ∂X. Sometimes we call these operators *b-pseudodifferential operators* in order to distinguish them from pseudodifferential operators on a manifold without boundary we defined in the preveious section. Throughout this section all manifolds are with boundary unless specified.

Our construction begins with a Lie algebra $\mathcal{V}_b(X)$ consisting of all C^∞ vector fields on X that are tangent to ∂X. Because X is a manifold with boundary, locally it is diffeomorphic to an open subset U of $[0,\infty)\times\mathbb{R}^{n-1}$. In terms of the corresponding local coordinates (x,y) with x a boundary defining function, it is easy to verify that $\mathcal{V}_b(X)$ is spanned over $C^\infty(X)$ by $x\partial_x$, ∂_y.

Indeed we can define the *b-tangent bundle* of X, denoted by ${}^bT(X)$, so that $\mathcal{V}_b(X)$ is the space of all smooth sections of ${}^bT(X)$. The *b-cotangent bundle* ${}^bT^*(X)$ of X is simply defined as the dual vector bundle of ${}^bT(X)$.

There is a natural map $\chi :\ T^*(X) \to {}^bT^*(X)$, which in our local coordinates takes the form

$$\chi(x,y,\xi,\zeta) = (x,y,x\xi,\zeta) \tag{41}$$

where ξ and ζ are dual variables of x and y in $T^*(X)$. Note that χ is a C^∞ map, but on ∂X it is not a diffeomorphism.

Being a C^∞ section of ${}^bT(X)$, every $V\in\mathcal{V}_b(X)$ defines a linear functional on each fiber of ${}^bT^*(X)$. This is the principal symbol map σ_b of $\mathcal{V}_b(X)$ to the set of C^∞ fiber-linear functions on ${}^bT^*(X)$.

Now consider the set $\mathrm{Diff}_b^m(X)$ whose elements are finite sums of tems of the form $aV_1V_2\cdots V_l$, $l\le m$, $V_j\in\mathcal{V}_b(X)$, $a\in C^\infty(X)$. Then $\mathrm{Diff}_b^\infty(X)=\bigcup_m \mathrm{Diff}_b^m(X)$ is a filtered algebra over $C^\infty(X)$. Defining $\sigma_{b,0}(a)=\pi^*(a)$ for $a\in C^\infty(X)$, π being the bundle projection ${}^bT^*(X)\to X$, σ_b extends to a map $\sigma_{b,m}$ of $\mathrm{Diff}_b^m(X)$ to the set of C^∞ fiber-homogeneous polynomials of degree m on ${}^bT^*(X)$. We can show that this is a filtered ring-homomorphism in the sense that

$$\sigma_{b,m+m'}(AB) = \sigma_{b,m}(A)\sigma_{b,m'}(B) \qquad A \in \mathrm{Diff}_b^m(X),\ B \in \mathrm{Diff}_b^{m'}(X). \tag{42}$$

The spaces $\mathrm{Diff}_b^m(X)$ extend to the spaces of b-pseudodifferential operators. The space of b-pseudodifferential operators of order m is denoted by $\Psi_b^m(X)$. They have the algebraic properties analogous to $\Psi^m(X)$ as the following theorem states.

Theorem 3.1.1. *Suppose X is a manifold (with boundary) of dimension n.*

(i) *The spaces $\Psi_b^m(X)$ increases with m. That is, $\Psi_b^m(X) \subset \Psi_b^{m'}(X)$ whenever $m \leq m'$.*

(ii) *If $A \in \Psi_b^m(X)$ then $A^* \in \Psi_b^m(X)$.*

(iii) *$\Psi_b^m(X) \cdot \Psi_b^{m'}(X) \subset \Psi_b^{m+m'}(X)$. That is, if $A \in \Psi_b^{m'}(X)$, $B \in \Psi_b^{m'}(X)$ then $AB \in \Psi_b^{m+m'}(X)$.*

Just as in (iii)′ of Theorem 2.1.6, we can combine (ii) and (iii) into a single statement.

(iii)′ $\Psi^\infty(X) = \bigcup_m \Psi^m(X)$ is an order-filtered $*$-algebra.

L^2 boundedness is also valid for b-pseudodifferential operators.

Theorem 3.1.2. *Suppose that X is a compact manifold. Every $A \in \Psi_b^0(X)$ extends from $\mathcal{S}(X)$ by continuity to define a continuous linear map on $L^2(X)$.*

The principal symbol map, which is already defined above for $\mathrm{Diff}_b^m(X)$, extends to a map σ_m on $\Psi_b^m(X)$. The extended map is still a filtered ring-homomorphism as (42) holds for b-pseudodifferential operators.

$$\sigma_{b,m+m'}(AB) = \sigma_{b,m}(A)\sigma_{b,m'}(B) \qquad A \in \Psi_b^m(X),\ B \in \Psi_b^{m'}(X). \tag{43}$$

Now we can define the *b-wavefront set* of distributions. For simplicity we define the b-wavefront set for $u \in L_g^2(X)$ where g is a Riemannian metric. The complete definition of the b-wavefront set involves some advanced concepts that are not of our interest. The complete definition can be found in [] with all the necessary concepts explained.

Definition 3.1.3. Suppose that $u \in L_g^2(X)$. We say that $(x, y, \sigma, \xi) \in {}^bT^*(X)$ is not in $WF_b(u)$ if there is an $A \in \Psi_b^0(X)$ such that A is elliptic at (x, y, σ, ξ) and $QAu \in L^2(X)$ for all $Q \in \mathrm{Diff}_b(X)$.

The notion of ellipticity of b-pseudodifferential operators can be developed in an analogous way to that of pseudodifferential operators. Note that this definition is stated in a parallel manner of our alternative definition of the wavefront set in Theorem 2.2.4.

The b-operator wavefront set can also be defined analogous to the operator wavefront set. However, because the definition involves several complicated concepts, we will not give any concrete descrption here.

3.2 Bicharacteristic geometry

With the b-calculi preliminaries, we can now turn to the bicharacteristics. Throughout this section, we will concentrate only on the wave operator.

Recall that if $A \in \Psi^m(X)$ with the principal symbol $\sigma_m(A) = a$, the characteristic set $\Sigma(A)$ is a subset of $T^*(X)$ and the bicharacteristics are integral curves of the Hamilton vector field of a inside $\Sigma(A)$.

Definitoin 3.2.1. Suppose that $A \in \Psi_b^m(X)$ with the principal symbol $\sigma_m(A) = a$. Define $\chi : T^*(X) \to {}^bT^*(X)$ as in (41). The *compressed characteristic set* $\dot{\Sigma}(A) \subset {}^bT^*(X)$ is defined by $\dot{\Sigma}(A) = \chi^{-1}(\Sigma(A))$. *Generalized broken bicharacteristics* are continuous maps $\gamma : I \to \dot{\Sigma}(A)$, where I is an interval, satisfying

$$\liminf_{s \to s_0} \frac{(f \circ \gamma)(s) - (f \circ \gamma)(s_0)}{s - s_0} \geq \inf_{\chi^{-1}(\gamma(s_0)) \cap \Sigma(A)} H_a(\chi^* f)(b) \tag{44}$$

for all $f \in C^\infty({}^bT^*(X))$ real valued.

The requirement (44) for is indeed very natural. For a manifold without boundary X, we have defined bicharacteristics as integral curves of the Hamilton vector field of the principal symbol a in the characteristic set. Thus if γ is an bicharacteristic segment over X, then for all $f \in C^\infty(T^*(X))$, the derivative of f along *gamma* at s_0, i.e. $\lim_{s \to s_0} \frac{(f \circ \gamma)(s) - (f \circ \gamma)(s_0)}{s - s_0}$, is equal to $H_a f(\gamma(s_0))$. When we return to the manifold with boundary X, H_a is a vector field on $T^*(X)$ while the image of γ lies in ${}^bT^*(X)$. Because χ is not one-to-one even if restricted to $\Sigma(A)$, the preimage of $\gamma(s_0)$ under χ often contains many points. Therefore f is possibly not differentiable along γ. Still we can expect bounds for the $\liminf$ of the difference quotients by taking the infimum of $H_a(\pi^* f)(b)$ over $\chi^{-1}(\gamma(s_0)) \cap \Sigma(A)$.

For the rest of this section, we will narrow our focus to the wave operator $P = D_t^2 - \Delta_g$ where g is a Riemannian metric on a manifold with boundary X. With the local coordinates (x, y) on X with x a boundary defining function, we will correspondingly write the local coordinates on $T^*(X)$ and ${}^bT^*(X)$ by (x, y, ξ, ζ) and (x, y, σ, ζ). Sometimes we will consider a manifold $Y = X \times \mathbb{R}$ with the local coordinates given by (x, y, t). The dual variable of t on $T^*(Y)$ and ${}^bT^*(Y)$ will be denoted by τ.

The generalized broken bicharacteristics for the wave operator P can be discribed more consisely. First we divide the compressed characteristic set $\dot{\Sigma}(P)$ into the following two subsets: the *glancing set* $\mathcal{G}$, the set of points in $\dot{\Sigma}(P)$ whose preimage under $\hat{\chi} = \chi|_\Sigma(P)$ is a single point set, and its complement, the *hyperbolic set* $\mathcal{H}$. Then we write the metric g in the form

$$g(x, y, \xi, \zeta) = \sum_{i,j} A_{ij}(x, y)\xi_i\zeta_j + \sum i, j2C_{ij}(x, y)\xi_i\zeta_j + \sum_{i,j} B_{ij}(x, y)\xi_i\zeta_j \tag{45}$$

with A, B, C smooth. The coordinates on X can be further adjusted so that $C(0, y) = 0$, and with this adjustment we have

$$a|_{x=0}(y, \xi, \zeta, \tau) = \tau^2 - \xi \cdot A(y)\xi - \zeta \cdot B(y)\zeta$$

on $T^*(Y) \setminus o$ with A, B positive definite matrices depending smoothly on y. Thus, with $\mathcal{U} = \{(x, y, t, \sigma, \zeta, \tau) \in {}^bT^*(Y) \setminus o : \ x = 0\}$, we have

$$\begin{aligned} \mathcal{G} \cap \mathcal{U} &= \{(0, y, t, 0, \zeta, \tau) \in {}^bT^*(Y) : \ \tau^2 = \zeta \cdot B(y)\zeta, \ (\zeta, \tau) \neq 0\}, \\ \mathcal{G} \cap \mathcal{U} &= \{(0, y, t, 0, \zeta, \tau) \in {}^bT^*(Y) : \ \tau^2 > \zeta \cdot B(y)\zeta, \ (\zeta, \tau) \neq 0\}. \end{aligned}$$

Note that $\dot{\Sigma}(P)$ is disjoint from all points $(x, y, t, \sigma, \zeta, \tau) \in {}^bT^*(Y)$ with $x = 0$ at which either $\sigma \neq 0$ or $\tau^2 < \zeta \cdot B(y)\zeta$.

The following theorem provides a concrete description of the generalized broken bicharacteristics of the wave operator.

Theorem 3.2.2. [1, Lemma 7] *Suppose γ is a generalized broken bicharacteristic. We let $\mathcal{G}$ and $\mathcal{H}$ denote the glancing set and the hyperbolic set contained in $\dot{\Sigma}(P)$ where P is the wave operator with the principal symbol p. We also define the map $\chi : \ T^*(X) \to {}^bT^*(X)$ as in* (41).

(i) *If $\gamma(s_0) \in \mathcal{G}$, let q_0 be the unique point in the preimage of $\gamma(s_0)$ under $\hat{\chi} = \chi|_{\Sigma}(P)$. Then for all $f \in C^\infty({}^bT^*(X))$ real valued, $f \circ \gamma$ is differentiable at s_0, and*

$$\frac{\mathrm{d}(f \circ \gamma)}{\mathrm{d}s}|_{s=s_0} = H_p \chi^* f(q_0).$$

(ii) *If $\gamma(s_0) \in \mathcal{H}$, lying over the boundary given in local coordinates by $x = 0$, then there exists $\epsilon > 0$ such that $x(\gamma(s)) = 0$ for $s \in (s_0 - \epsilon, s_0 + \epsilon)$ if and only if $s = s_0$. That is, γ does not meet the corner in a punctured neighborhood of s_0.*

3.3 Propagation of Singularities

We are now ready to state the main theorem.

For a manifold X with boundary ∂X, we define $H_0^1(X)$ as the completion of $C_0^\infty(\mathrm{int}(X))$ in the norm

$$\|u\|^2_{H_0^1(X)} = \|\mathrm{d}u\|^2_{L^2(X)} + \|u\|^2_{L^2(X)}$$

whose elements restrict as 0 to ∂X. Thus all $u \in H^2_{0,loc}(X)$ satisfies the Dirichlet boundary conditions.

Theorem 3.3.1.(cf.[1, Theorem 9]) *Suppose $A \in \Psi_b^m(X)$, $u \in H^1_{0,loc}(X)$ and $Au = 0$.*

(i) *$WF_b(u) \subset \dot{\Sigma}(A)$*

(ii) *$WF_b(u)$ is a union of maximally extended generalized broken bicharacteristics in $\dot{\Sigma}(A)$. That is, if $s \in WF_b(u)$, then so is the whole bicharacteristic segment through s.*

This theorem can be stated in a completely microlocal manner. The theorem also holds for Neumann boundary conditions.

It was proved in the real analytic setting by Lebeau [], and in the C^∞ setting with C^∞ boundaries by Melrose, Sjöstrand and Taylor []. The main ideas of the proof are the positive commutator estimates, b-microlocalization, and the Dirichlet form.

Here we sketch the positive commutator proof in the boundarylesss setting. For $A \in \Psi^\infty(X)$, we want to construct $B \in \Psi^\infty(X)$ such that $i[B^*B, A]$ is positive, modulo terms we have control. Note that

$$\begin{aligned}\langle i[B^*B, A]u, u\rangle &= \langle iB^*BAu, u\rangle - \langle iAB^*Bu, u\rangle \\ &= \langle iB^*BAu, u\rangle + \langle u, iB^*BAu\rangle,\end{aligned}$$

and the PDE gives information about Au. Now if $i[B^*B, A] = C^*C + E + F$ where F is lower order, hence negligible, we deduce that $\|Cu\|^2$ can be controlled provided we have information about $\langle Eu, u\rangle$, i.e. about u microlocally on $WF'(E)$.

In our positive commutator estimates we therefore assume that $WF(u)$ is disjoint from $WF'(E)$ and propagate the regularity of u to conclude that $WF(u)$ is disjoint from the elliptic set of C. We accomplish this with an iterative argument showing that u is in the Sobolev space $H^m(X)$ on the elliptic set of C, provided that it is in $H^{m-\frac{1}{2}}(X)$ in a neighborhood, that $WF(u)$ is disjoint from $WF'(E)$ and that $Au = 0$. Therefore we need $C \in \Psi^m(X)$, for then the L^2 norm of Cu is a microlocal H^m norm of u, and so we eventually need $B \in \Psi^{m-\frac{1}{2}}(X)$.

The equation $i[B^*B, A] = C^*C + E + F$ is indded a condition on the principal symble b of B, namely that

$$H_a b = -c^2 + e$$

where a, c and e are the principal symbols of A, C and E respectively. We want b to be decreasing along bicharacteristics except in a region in which we have information about u. We also want b to have small support.

In our simple setting, H_a can locally be made into a constant vector field $\frac{\partial}{\partial q_1}$ by a change of coordiates on $S^*(X) = (T^*(X) \setminus o)/\mathbb{R}^+$. So we work with homogeneous degree zero functions $q = (q_1, \tilde{q})$ on $T^*(X) \setminus o$ which give coordinates on $S^*(X)$. In such coordiates the construction of b as a C^∞ function on $S^*(X)$ is straightforward which is then regarded as a homogeneous function on $T^*(X) \setminus o$. More precisely, we can take b to be the product of a compactly supported function θ_1 of q_1, on the derivative of which we have sign conditions, and an arbitrary compactly supported non-negative function θ_2 of $\tilde{q}$. Here the compact support should be small enough so that supp a is a subset of the region of validity of the local coordiates.

Here supp e corresponds to the region where θ_1' is positive, or equivalently, where $H_a b = \theta_1'\theta_2$ is positive. Note that the positive commutator estimates gives us control of a solution of $Au = 0$ over the rest of supp a provided we have control of u over supp e. Because we can choose θ_2 to have its support in an arbitrary small set, we conclude that the control of u propagates along the bicharacteristics.

4 Acknowledgement

I would like to thank Professor Andras Vasy for his guidance on this honors thesis and for his infinite patience in explaining important concepts, refining my arguments and revising my draft, for which I am much more grateful than I can express in this short note.

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