

# HEARING THE SHAPE OF A DRUM

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## 1. INTRODUCTION

In 1966, Mark Kac published an article in American Mathematical Monthly discussing a problem that had been suggested to him by Salomon Bochner; the problem was, given the eigenvalue spectrum of the laplacian operator on a certain domain, can one reconstruct the domain? To the layman, this question may sound incomprehensible, but it turns out to have an easily understood physical meaning. This problem now bears the title “can one hear the shape of a drum?”

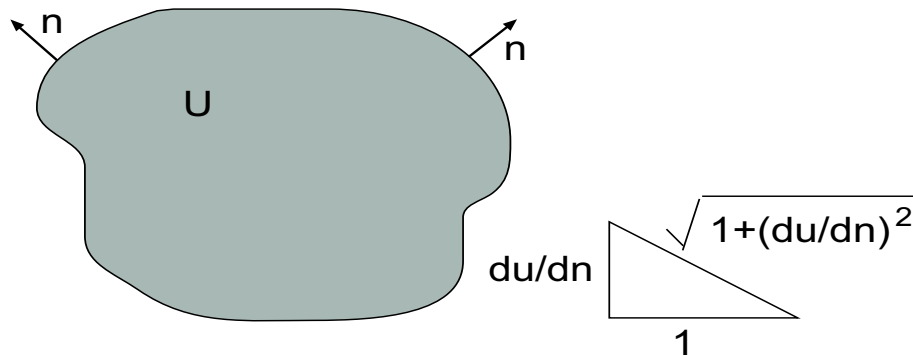
We will state our problem more precisely. First, we consider what it means to hear a sound from a two-dimensional drum and provide motivation for the wave equation. We wish to consider a flexible membrane stretched over a certain domain, with the boundary fixed at 0. The membrane will have small vertical displacement, but will not have any horizontal displacement. Let  $u(x, y, t)$  be the vertical displacement of the membrane over the domain, which we call  $U$ , and let the boundary of this domain be  $\partial U$ .

We want to use Newton’s second law,  $F = ma$ . We have the equation

$$\iint_U \rho u_{tt} \, dx dy = ma.$$

Now the gradient of  $u$  in the direction of the normal vector  $\vec{n}$  is  $\frac{\partial u}{\partial n}$ . Thus, the transverse tension at a certain point on the boundary is approximately

$$\frac{T \frac{\partial u}{\partial n}}{\sqrt{1 + (\frac{\partial u}{\partial n})^2}}$$



from the triangle in the figure above, and by Taylor series approximation, we have  $\sqrt{1 + (\frac{\partial u}{\partial n})^2} \approx 1$ , so the transverse tension is  $T \frac{\partial u}{\partial n}$ . And the horizontal tension is

$$\frac{T}{\sqrt{1 + (\frac{\partial u}{\partial n})^2}}.$$

But the horizontal displacement is 0, so  $T$  must be constant. Then

$$\int_{\partial U} T \frac{\partial u}{\partial n} ds = F = ma = \int \int_U \rho u_{tt} dx dy.$$

Now, by Green's Theorem, we get

$$\int \int_U \operatorname{div}(T \nabla u) ds = \int \int_U \rho u_{tt} dx dy.$$

By a vanishing theorem (see A.1, Second Vanishing Theorem in [1]), we have  $\operatorname{div}(T \nabla u) = \rho u_{tt}$ , and since  $T$  is constant,

$$u_{tt} = \frac{T}{\rho} \Delta u,$$

the wave equation.

Now let  $c = \sqrt{\frac{T}{\rho}}$ . The wave equation with Dirichlet boundary conditions is

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & \text{in } U \\ u = 0 & \text{in } \partial U. \end{cases}$$

Let  $c = 1$ . Now if we use the method of separation of variables to look for a solution of the form

$$u(x, t) = X(x)T(t).$$

Then by the wave equation,  $XT'' - \Delta XT = 0$ , so we will look for solutions of

$$-\frac{T''}{T} = -\frac{\Delta X}{X} = \lambda$$

for constant  $\lambda$ . Then we get the separated system

$$\begin{cases} T(t) = Ce^{i\sqrt{\lambda}t} & \text{if } t \in \mathbb{R} \\ \Delta X(x) + \lambda X(x) = 0 & \text{if } x \in U \\ X(x) = 0 & \text{if } x \in \partial U. \end{cases}$$

Thus, finding the solutions to the wave equation reduces to finding the eigenvalues and eigenfunctions of the laplacian operator  $\Delta$ . Now if we are given

$$\Sigma = \{\lambda \mid \Delta X(x) + \lambda X(x) = 0 \text{ for some } X\},$$

the set of all eigenvalues of  $\Delta$ , we would like to know if we can reconstruct the domain  $U$ . We call  $\Sigma$  the eigenvalue spectrum and say that two domains are isospectral if they have the same eigenvalue spectrum. If two domains are congruent, we say the domains are isometric.

What kind of structure does the eigenvalue spectrum have? In order to understand our problem, it will help to see how the eigenvalues are placed, whether they

form a discrete sequence or whether there are eigen-intervals, intervals of the real line in which every value is an eigenvalue. We claim that this cannot be the case.

In sections 2 to 4 of this paper, we will show that

$$\Sigma = \{\lambda_n\}_{n=1}^{\infty}$$

for some sequence of real numbers  $\{\lambda_n\}_{n=1}^{\infty}$ ; that is, the eigenvalue spectrum is discrete. We start off in section 2 by giving a proof of the Spectral Theorem, which states that a compact, self-adjoint operator (defined later) has a discrete eigenvalue spectrum. We cannot apply the Spectral Theorem directly to the laplacian operator since the laplacian is not compact. However, the inverse of the laplacian is compact, so we can still use the Spectral Theorem successfully by applying it to this inverse.

The subject of sections 3 and 4 is to show that we can indeed use the inverse of the laplacian in this manner. We will prove these results more generally for elliptic operators, a class of operators to which the inverse of the laplacian belongs. In section 3, we obtain compactness on certain spaces used to construct the inverse of an elliptic operator, by proving the Rellich-Kondrachov Compactness Theorem. In section 4 we will see that the inverse of an elliptic operator is compact. We first obtain conditions on which we can find a weak solution of an operator, given by the Lax-Milgram Theorem. These conditions can be satisfied with any given elliptic operator and we thus obtain a weak solution to the corresponding partial differential equation. Applying the Spectral Theorem to the inverse of our elliptic operator, there is a nice relation between the eigenvalues of the inverse and the eigenvalues of the original operator. This relation gives the desired result, that the eigenvalue spectrum is discrete.

Finally, in section 5, we will turn our attention to the problem of hearing the shape of a drum and what can be determined from the eigenvalue spectrum. In particular, we will see that the eigenvalue spectrum determines the area and perimeter of the domain, as well as whether the domain is multiply connected. But we will see that in general the spectrum cannot determine the original domain, and we give a few examples of isospectral domains and methods that can be used to construct them, concluding with an example of isospectral domains where we can explicitly compute the eigenvalue spectra.

## 2. THE SPECTRAL THEOREM FOR COMPACT SELF-ADJOINT OPERATORS

First, we give two definitions on special operators.

**Definition 1.** An operator  $A$  is self-adjoint if  $A = A^*$ .

**Definition 2.** A linear operator  $T$  is compact if the image of the unit ball under  $T$  has compact closure in  $\mathcal{H}$ .

The Spectral Theorem is useful for answering many questions about the spectrum of compact, self-adjoint operators. In fact, the Spectral Theorem not only applies to compact, self-adjoint operators, but also compact, normal operators as well (for further treatment of the Spectral Theorem, see [7]). We now establish a few results.

**Lemma 3.** *Let  $A$  be a self-adjoint operator on a Hilbert space  $\mathcal{H}$ . Then there exists a sequence of unit vectors  $\{h_n\}_{n=1}^{\infty}$  such that  $|\langle Ah_n, h_n \rangle| \rightarrow \|A\|$  as  $n \rightarrow \infty$ .*

*Proof.* It suffices to show that  $\sup_{h \in \mathcal{H}, \|h\|=1} |\langle Ah, h \rangle| = \|A\|$ .

First we show that  $\sup_{h \in \mathcal{H}, \|h\|=1} |\langle Ah, h \rangle| \leq \|A\|$ . By the Cauchy-Schwarz inequality, for  $\|h\| = 1$ ,

$$\begin{aligned} |\langle Ah, h \rangle| &\leq \|Ah\| \|h\| \\ &= \frac{\|Ah\|}{\|h\|} \\ &\leq \|A\|. \end{aligned}$$

Taking the supremum, we get  $\sup_{h \in \mathcal{H}, \|h\|=1} |\langle Ah, h \rangle| \leq \|A\|$ .

Now we show that  $\sup_{h \in \mathcal{H}, \|h\|=1} |\langle Ah, h \rangle| \geq \|A\|$ . We want to show that for any  $\|f\| = \|g\| = 1$ ,

$$\operatorname{Re} \langle Af, g \rangle \leq \sup_{h \in \mathcal{H}, \|h\|=1} \langle Ah, h \rangle. \quad (1)$$

Then if  $\langle Af, g \rangle = e^{i\theta} |\langle Af, g \rangle|$ , we can substitute  $e^{-i\theta} f$  for  $f$  in (1) to get

$$\begin{aligned} \sup_{h \in \mathcal{H}, \|h\|=1} |\langle Ah, h \rangle| &\geq \operatorname{Re} \langle Ae^{-i\theta} f, g \rangle \\ &= \operatorname{Re}(e^{-i\theta} \langle Af, g \rangle) \\ &= |\langle Af, g \rangle|, \end{aligned}$$

and letting  $g = \frac{Af}{\|Af\|}$ ,

$$\begin{aligned} |\langle Af, g \rangle| &= \left| \langle Af, \frac{Af}{\|Af\|} \rangle \right| \\ &= \|Af\|. \end{aligned}$$

Taking the supremum over  $\|f\| = 1$ , we have  $\sup_{h \in \mathcal{H}, \|h\|=1} |\langle Ah, h \rangle| \geq \|A\|$ .

It remains to show that  $\operatorname{Re} \langle Af, g \rangle \leq \sup_{h \in \mathcal{H}, \|h\|=1} \langle Ah, h \rangle$ .

We have

$$\langle A(f+g), f+g \rangle = \langle Af, f \rangle + \langle Ag, g \rangle + \langle Af, g \rangle + \langle Ag, f \rangle$$

and

$$\langle A(f-g), f-g \rangle = \langle Af, f \rangle + \langle Ag, g \rangle - \langle Af, g \rangle - \langle Ag, f \rangle.$$

Subtracting, we have

$$\begin{aligned} \langle A(f+g), f+g \rangle - \langle A(f-g), f-g \rangle &= 2(\langle Af, g \rangle + \langle Ag, f \rangle) \\ &= 2(\langle Af, g \rangle + \langle g, Af \rangle) \\ &= 2(\langle Af, g \rangle + \overline{\langle Af, g \rangle}) \\ &= 4\operatorname{Re} \langle Af, g \rangle. \end{aligned}$$

Then

$$\begin{aligned}
4\operatorname{Re}\langle Af, g \rangle &= \langle A(f+g), f+g \rangle - \langle A(f-g), f-g \rangle \\
&= \left\langle A \left( \frac{f+g}{\|f+g\|} \right), \frac{f+g}{\|f+g\|} \right\rangle \|f+g\|^2 + \left\langle A \left( \frac{f-g}{\|f-g\|} \right), \frac{f-g}{\|f-g\|} \right\rangle \|f-g\|^2 \\
&\leq \sup_{h \in \mathcal{H}, \|h\|=1} |\langle Ah, h \rangle| (\|f+g\|^2 + \|f-g\|^2) \\
&= \sup_{h \in \mathcal{H}, \|h\|=1} |\langle Ah, h \rangle| (2\|f\|^2 + 2\|g\|^2) \text{ (parallelogram law)} \\
&= 4 \sup_{h \in \mathcal{H}, \|h\|=1} |\langle Ah, h \rangle|.
\end{aligned}$$

Then  $\operatorname{Re}\langle Af, g \rangle \leq \sup_{h \in \mathcal{H}, \|h\|=1} \langle Ah, h \rangle$  as needed.  $\square$

**Lemma 4.** *Let  $A$  be a self-adjoint operator. If  $\lambda$  is an eigenvalue of  $A$ ,  $\lambda$  is real.*

*Proof.* Since  $\lambda$  is an eigenvalue,  $Ah = \lambda h$  for some nonzero  $h \in \mathcal{H}$ . Then

$$\begin{aligned}
\lambda \langle h, h \rangle &= \langle Ah, h \rangle \\
&= \langle h, Ah \rangle \\
&= \bar{\lambda} \langle h, h \rangle,
\end{aligned}$$

so  $\lambda = \bar{\lambda}$ , which implies  $\lambda$  is real.  $\square$

**Lemma 5.** *Let  $B$  be a compact operator on a Hilbert space  $\mathcal{H}$ . If there exists a sequence  $\{h_n\}_{n=1}^\infty$  and a real number  $\lambda$  such that  $\|(B - \lambda)h_n\| \rightarrow 0$  as  $n \rightarrow \infty$ , then  $\lambda$  is an eigenvalue of  $B$ .*

*Proof.* Since  $B$  is compact, there exists a subsequence  $\{h_{n_i}\}_{i=1}^\infty$  of  $\{h_n\}_{n=1}^\infty$  such that  $\|Bh_{n_i} - h\| \rightarrow 0$  for some  $h \in \mathcal{H}$ . Now  $\lambda h_{n_i} = (\lambda - B)h_{n_i} + Bh_{n_i} \rightarrow 0 + h = h$ . But

$$\begin{aligned}
\|B(\lambda h_{n_i} - h)\| &= \left\| B \left( \frac{\lambda h_{n_i} - h}{\|\lambda h_{n_i} - h\|} \right) \right\| \|\lambda h_{n_i} - h\| \\
&\leq \|B\| \|\lambda h_{n_i} - h\|,
\end{aligned}$$

which converges to 0 as  $i \rightarrow \infty$ . Then  $\|Bh_{n_i} - h\| \rightarrow 0$  and  $\|Bh_{n_i} - \lambda^{-1}Bh\| \rightarrow 0$ , so  $h = \lambda^{-1}Bh$ . Therefore,  $\lambda$  is an eigenvalue.  $\square$

**Lemma 6.** *Let  $T$  be a compact, self-adjoint operator on a Hilbert space  $\mathcal{H}$ . Then  $\lambda$  is an eigenvalue of  $T$ , where  $\lambda = \|T\|$  or  $-\|T\|$ .*

*Proof.* By Lemma 3, we can find a sequence of unit vectors  $\{h_n\}_{n=1}^\infty$  such that  $\langle Th_n, h_n \rangle \rightarrow \lambda$  as  $n \rightarrow \infty$ , where  $\lambda = \|T\|$  or  $-\|T\|$ . Then

$$\begin{aligned}
\|(T - \lambda)h_n\|^2 &= \|Th_n\|^2 + \lambda^2 \|h_n\|^2 - 2\lambda \langle Th_n, h_n \rangle \\
&\leq \|T\|^2 + \lambda^2 - 2\lambda \langle Th_n, h_n \rangle \\
&\leq 2\lambda^2 - 2\lambda \langle Th_n, h_n \rangle,
\end{aligned}$$

which converges to 0 as  $n \rightarrow \infty$ . By Lemma 5,  $\lambda$  is an eigenvalue of  $T$ .

□

Now we state the Spectral Theorem and give a proof.

**Theorem 7.** *Let  $T$  be a compact, self-adjoint operator on a Hilbert space  $\mathcal{H}$ . Then there exists a sequence  $\{\lambda_n\}_{n=1}^{\infty}$  of eigenvalues of  $T$  such that*

- (1)  $|\lambda_i| \geq |\lambda_{i+1}|$  for  $i = 1, 2, \dots$ ,
- (2) if  $P_i$  = the projection of  $\mathcal{H}$  onto  $\mathcal{E}_i = \ker(T - \lambda_i)$ , then  $P_n P_m = P_m P_n = 0$  for  $n \neq m$ ,
- (3)  $T = \sum_{i=1}^{\infty} \lambda_i P_i$  in the sense that  $\|T - \sum_{i=1}^n \lambda_i P_i\| \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof.* By Lemma 6,  $\lambda_1$  is an eigenvalue of  $T$ , where  $\lambda_1 = \|T\|$  or  $-\|T\|$ . We will show that  $\mathcal{E}_1 = \ker(T - \lambda_1)$  reduces  $T$  so that we can define an operator that is the restriction of  $T$  to  $\mathcal{E}_1^\perp$ . If  $x \in \mathcal{E}_1$ ,

$$(T - \lambda_1)Tx = (T - \lambda_1)\lambda_1 x = 0,$$

so  $Tx \in \mathcal{E}_1$ . Now if  $x \in \mathcal{E}_1^\perp$ ,  $\langle x, y \rangle = 0$  for some  $y \in \mathcal{E}_1$ . Then

$$\langle Tx, y \rangle = \langle x, Ty \rangle = 0$$

since  $Ty \in \mathcal{E}_1$ . Therefore,  $\mathcal{E}_1 = \ker(T - \lambda_1)$  reduces  $T$ .

Thus, we can define

$$T_2 = T|_{\mathcal{E}_1^\perp}.$$

We will show that  $T_2$  is a compact, self-adjoint operator. Let  $\{h_n\}_{n=1}^{\infty}$  be a sequence in  $\mathcal{E}_1^\perp$ . Since  $T$  is compact, there exists a subsequence  $\{h_{n_i}\}_{i=1}^{\infty}$  and an  $h \in \text{cl}(\mathcal{E}_1^\perp)$  such that  $Th_{n_i} \rightarrow h$ . And  $T_2$  is the restriction of  $T$  to  $\mathcal{E}_1^\perp$ , so  $T_2 h_{n_i} \rightarrow h$ . Therefore,  $T_2$  is compact. Now  $\langle Tx, y \rangle = \langle x, Ty \rangle$  for  $x, y \in \mathcal{E}_1^\perp$ , so  $\langle T_2 x, y \rangle = \langle x, T_2 y \rangle$ . Therefore,  $T_2$  is self-adjoint.

By Lemma 6,  $\lambda_2$  is an eigenvalue of  $T_2$ , where  $\lambda_2 = \|T_2\|$  or  $-\|T_2\|$ . And  $|\lambda_1| \geq |\lambda_2|$  since  $\|T\| \geq \|T_2\|$ .

Now we extend this result to obtain a distinct sequence of eigenvalues converging to 0. We proceed by induction. Suppose we can define

$$T_i = T|_{(\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_{i-1})^\perp} \text{ and } \mathcal{E}_i = \ker(T_i - \lambda_i)$$

for  $i = 3, 4, \dots, n$  recursively and that  $T_i$ ,  $i = 3, 4, \dots, n$ , are compact, self-adjoint operators. Then by Lemma 6 and the fact that

$$(\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_i)^\perp \subseteq (\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_{i-1})^\perp,$$

we would have a sequence of eigenvalues  $\{\lambda_i\}_{i=1}^n$  such that  $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$ .

Note that this definition for  $\mathcal{E}_i$  is slightly different than the one given in the statement of the theorem. We will later prove that the two definitions are equivalent.

We now show that  $\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n$  reduces  $T$ . If  $x \in \mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n$ ,  $x = \sum_{i=1}^n x_i$  for some  $x_i \in \mathcal{E}_i$ ,  $i = 1, \dots, n$ . Then

$$Tx = \sum_{i=1}^n Tx_i = \sum_{i=1}^n \lambda_i x_i \text{ and } (T_j - \lambda_j)\lambda_j x_j = 0$$

for  $j = 1, 2, \dots, n$ , so  $Tx \in \mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n$ . Now suppose  $x \in (\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n)^\perp$ . Then  $\langle x, y \rangle = 0$  for  $y \in \mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n$ . But

$$\langle Tx, y \rangle = \langle x, Ty \rangle = 0$$

since  $Ty \in \mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n$ . Therefore,  $Tx \in (\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n)^\perp$ , and hence  $\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n$  reduces  $T$ .

Therefore, we can define

$$T_{n+1} = T|_{(\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n)^\perp}.$$

Now we show that  $T_{n+1}$  is a compact, self-adjoint operator. Let  $\{h_n\}_{n=1}^\infty$  be a sequence in  $(\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n)^\perp$ . Since  $T$  is compact, there exists a subsequence  $\{h_{n_i}\}_{i=1}^\infty$  and  $h \in cl((\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n)^\perp)$  such that  $Th_{n_i} \rightarrow h$ . But  $T_{n+1}$  is the restriction of  $T$  to  $(\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n)^\perp$ , so  $T_{n+1}h_{n_i} \rightarrow h$ . Thus,  $T_{n+1}$  is compact. Now  $\langle Tx, y \rangle = \langle x, Ty \rangle$  for  $x, y \in (\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n)^\perp$ , so  $\langle T_{n+1}x, y \rangle = \langle x, T_{n+1}y \rangle$ . Therefore,  $T_{n+1}$  is self-adjoint.

By Lemma 6,  $\lambda_{n+1}$  is an eigenvalue of  $T_{n+1}$ , where  $\lambda_{n+1} = \|T_{n+1}\|$  or  $-\|T_{n+1}\|$ . And  $|\lambda_n| \geq |\lambda_{n+1}|$  since  $\|T_n\| \geq \|T_{n+1}\|$ .

By the Bounded Monotone Sequence Property,  $|\lambda_n| \rightarrow \alpha$  as  $n \rightarrow \infty$ , for some  $\alpha \in \mathbb{R}$ . We will show that  $\alpha = 0$ . First, we show that if  $\lambda_i = \lambda_j, i = j$ . Suppose  $\lambda_i = \lambda_j, i < j$ . Then

$$\mathcal{E}_j = \ker(T_j - \lambda_j) = \ker(T_j - \lambda_i) \subseteq \ker(T_i - \lambda_i) = \mathcal{E}_i.$$

But

$$\mathcal{E}_j = \ker(T_j - \lambda_j) \subseteq (\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_{j-1})^\perp \subseteq \mathcal{E}_i^\perp, \quad (2)$$

so  $\mathcal{E}_j = (0)$ , a contradiction that  $\lambda_j$  is an eigenvalue of  $T_j$ . Thus,  $\lambda_i \neq \lambda_j$  for  $i \neq j$ . Also note that by (2) we have shown for that  $\mathcal{E}_i \perp \mathcal{E}_j$  for  $i \neq j$ .

Also, if  $\lambda_i \neq \lambda_n$ , let  $h \in \ker(T - \lambda_n)$ . For  $x_i \in \mathcal{E}_i$ ,

$$\begin{aligned} \langle h, x_i \rangle &= \langle \lambda_n^{-1}Th, x_i \rangle \\ &= \lambda_n^{-1} \langle h, Tx_i \rangle \\ &= \lambda_n^{-1} \lambda_i \langle h, x_i \rangle, \end{aligned}$$

so  $(1 - \lambda_n^{-1}\lambda_i)\langle h, x_i \rangle = 0$  and since  $\lambda_n \neq \lambda_i, \langle h, x_i \rangle = 0$ . Thus,  $h \in (\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_{n-1})^\perp$ , so  $\ker(T - \lambda_n) \subseteq \mathcal{E}_n$ .  $\mathcal{E}_n \subseteq \ker(T - \lambda_n)$  follows trivially, so

$$\ker(T - \lambda_n) = \mathcal{E}_n$$

as we claimed earlier.

Now if  $\{e_n\}_{n=1}^\infty$  is a sequence of unit vectors such that  $e_n \in \mathcal{E}_n$ . Since  $T$  is compact, there exists a subsequence  $\{e_{n_i}\}_{i=1}^\infty$  such that  $\{Te_{n_i}\}_{i=1}^\infty$  converges. Then  $\{Te_{n_i}\}_{i=1}^\infty$  is also a Cauchy sequence. But

$$\begin{aligned} \|Te_{n_i} - Te_{n_j}\|^2 &= \|Te_{n_i}\|^2 + \|Te_{n_j}\|^2 \quad (\text{since } \mathcal{E}_{n_i} \perp \mathcal{E}_{n_j}) \\ &= \lambda_{n_i}^2 + \lambda_{n_j}^2 \\ &\geq 2\alpha^2. \end{aligned}$$

Since  $\{Te_{n_i}\}_{i=1}^\infty$  is a Cauchy sequence,  $\alpha = 0$ .

It remains to show that  $\|T - \sum_{i=1}^n \lambda_i P_i\| \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n$  reduces  $T$ ,

$$\|T - \sum_{i=1}^n \lambda_i P_i\| = \max(\|(T - \sum_{i=1}^n \lambda_i P_i)|_{(\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n)}\|, \|(T - \sum_{i=1}^n \lambda_i P_i)|_{(\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n)^\perp}\|).$$

But if  $x \in \mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n$ ,  $x = \sum_{i=1}^n x_i$  for some  $x_i \in \mathcal{E}_i$ . Then

$$\begin{aligned} (T - \sum_{j=1}^n \lambda_j P_j)x &= \sum_{i=1}^n (T - \sum_{j=1}^n \lambda_j P_j)x_i \\ &= \sum_{i=1}^n (T - \lambda_i)x_i \\ &= 0. \end{aligned}$$

And if  $x \in (\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n)^\perp$ ,

$$(T - \sum_{j=1}^n \lambda_j P_j)x = Tx - 0 = Tx.$$

Then we have

$$\begin{aligned} \|T - \sum_{i=1}^n \lambda_i P_i\| &= \max(0, \|T|(\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n)^\perp\|) \\ &= \|T_{n+1}\| \\ &= |\lambda_{n+1}|, \end{aligned}$$

which converges to 0 as  $n \rightarrow \infty$ . □

### 3. THE RELICH-KONDRACHOV COMPACTNESS THEOREM

Our goal in this section is to prove the Rellich-Kondrachov Compactness Theorem, which we will use to show that an operator from  $L^2(U)$  to  $L^2(U)$  that is bounded in  $H_0^1(U)$  must be compact.

We say that for Banach spaces  $X, Y$  such that  $X \subseteq Y$ ,  $X$  is compactly embedded in  $Y$ , written  $X \subset\subset Y$ , if

- (i) For all  $x \in X$ ,  $\|x\|_Y \leq C\|x\|_X$
- (ii) If  $\{x_n\}_{n=1}^\infty$  is a bounded sequence in  $X$ , it is precompact in  $Y$ .

First, we state a few results which we will assume without proof (for proofs of these results, see [6]).

Suppose  $1 \leq p < n$ . Let

$$p^* \equiv \frac{np}{n-p},$$

the Sobolev conjugate of  $p$ .

**Corollary to Poincaré's inequality.** *For  $1 \leq p \leq q$ , if  $U$  is bounded*

$$\|u\|_{L^p(U)} \leq C\|u\|_{L^q(U)}$$

for some constant  $C$ .

**Theorem 8.** *Let  $U$  be bounded and open in  $\mathbb{R}^n$  and suppose  $\partial U \in C^1(\mathbb{R}^{n-1})$ . Let  $1 \leq p < n$  and  $u \in W^{1,p}(U)$ . Then  $u \in L^{p^*}(U)$  and*

$$\|u\|_{L^{p^*}} \leq C\|u\|_{W^{1,p}(U)},$$

with  $C$  depending only on  $p, n$ , and  $U$ .

**Interpolation Inequality.** Let  $1 \leq s \leq r \leq t \leq \infty$  and  $\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}$ . If  $u \in L^s(U) \cap L^t(U)$ , then  $u \in L^r(U)$  and

$$\|u\|_{L^r(U)} \leq \|u\|_{L^s(U)}^\theta \|u\|_{L^t(U)}^{1-\theta}.$$

**Gagliardo-Nirenberg-Sobolev Inequality.** For  $1 \leq p < n$ , there exists a constant  $C$  such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)}$$

for all  $u \in C_c^1(\mathbb{R}^n)$ ,  $C$  depending only on  $p$  and  $n$ .

The next theorem gives a method for extending results in smooth functions to  $W^{k,p}(U)$ .

**Theorem 9.** Assume  $U$  is bounded,  $\partial U \in C^1(\mathbb{R}^{n-1})$  and that  $u \in W^{k,p}(U)$  for some  $1 \leq p < \infty$ . Then there exist functions  $u_m \in C^\infty(\text{cl}(U))$  such that

$$u_m \rightarrow u \text{ in } W^{k,p}(U).$$

**Extension Theorem.** Assume  $U$  is bounded and  $\partial U \in C^1(\mathbb{R}^{n-1})$ . For a bounded open set  $V$  such that  $U \subset\subset V$ , there exists a bounded linear operator  $E : W^{1,p}(U) \rightarrow W^{1,p}(\mathbb{R}^n)$  such that for each  $u \in W^{1,p}(U)$ ,

- (i)  $Eu = u$  a.e. in  $U$ ,
- (ii)  $Eu$  has support within  $V$ ,
- (iii)  $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(U)}$ ,

with  $C$  depending only on  $p$ ,  $U$  and  $V$ .

Now we state the Rellich-Kondrachov Compactness Theorem and give a proof.

**Rellich-Kondrachov Theorem.** Assume  $U$  is a bounded, open subset of  $\mathbb{R}^n$ , that  $\partial U$  is  $C^1$ , and  $1 \leq p < n$ . Then

$$W^{k,p}(U) \subset\subset L^p(U)$$

for each  $q$  such that  $1 \leq q < p^*$ .

*Proof.* We will first show  $\|u\|_{L^q(U)} \leq C \|u\|_{W^{1,p}(U)}$  for a constant  $C$ . Fix  $1 \leq q < p^*$ . By assumption  $U$  is bounded, so Theorem 8 implies

$$\|u\|_{L^{p^*}(U)} \leq C_1 \|u\|_{W^{1,p}(U)}$$

for some constant  $C_1$ . Now note that  $1 \leq q < p^*$ , so  $\frac{1}{q} = \frac{\theta}{p^*} + \frac{1-\theta}{1}$  for some  $0 \leq \theta \leq 1$ . Applying the Interpolation Inequality, we have

$$\begin{aligned} \|u\|_{L^q(U)} &\leq \|u\|_{L^{p^*}(U)}^\theta \|u\|_{L^1(U)}^{1-\theta} \\ &\leq \|u\|_{L^{p^*}(U)}^\theta \|u\|_{L^{p^*}(U)}^{1-\theta} \\ &= C_2 \|u\|_{L^{p^*}(U)} \end{aligned}$$

for some constant  $C_2$ , with the second inequality following from the Corollary to Poincaré's inequality.

Then  $\|u\|_{L^q(U)} \leq C \|u\|_{W^{1,p}(U)}$ . Thus, we have the first condition for compact embedding.

Now we show that for  $\{u_m\}_{m=1}^\infty$  a bounded sequence  $W^{1,p}(U)$ , there exists a sequence  $\{u_{m_i}\}_{i=1}^\infty$  converging in  $L^q(U)$ . By the Extension Theorem, we can assume without loss of generality that

$U = \mathbb{R}^n$ ,  $\{u_m\}$  has compact support in a bounded, open set  $V \subseteq \mathbb{R}^n$ , and that  $u_m$  is bounded in  $W^{1,p}(V)$ .

Now consider the functions  $u_m^\varepsilon \equiv \eta_\varepsilon * u_m$  for  $\varepsilon > 0$ ,  $m = 1, 2, \dots$ , where

$$\eta \equiv \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

and  $C = \frac{1}{\int_{-1}^1 \exp\left(\frac{1}{|x|^2-1}\right) dx}$  and  $\eta_\varepsilon \equiv \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right)$  (so that  $\int_{\mathbb{R}^n} \eta_\varepsilon dx = 1$ ).

We claim that  $u_m^\varepsilon$  has support in  $V$  for sufficiently small  $\varepsilon$ . We have that

$$u_m^\varepsilon(x) = \int_{B(0,\varepsilon)} \eta_\varepsilon(y) u_m(x-y) dy,$$

where  $B(x,r)$  is the ball centered at  $x$  with radius  $r$  in  $\mathbb{R}^n$ .

Then the support of  $u_m^\varepsilon(x)$  is contained in  $U^\varepsilon = \{x \in \mathbb{R}^n \mid \|x-y\| < \varepsilon, y \in U\}$ . But  $V$  is open, so we can choose  $\varepsilon$  small enough so that  $U^\varepsilon \subseteq V$ .

Now we show that  $u_m^\varepsilon \rightarrow u_m$  in  $L^q(V)$  as  $\varepsilon \rightarrow 0$ , uniformly with respect to  $m$ . First assume  $u_m \in C^\infty \cap W^{1,p}$ .

$$\begin{aligned} u_m^\varepsilon(x) - u_m(x) &= \int_{B(0,\varepsilon)} \eta_\varepsilon(y) u_m(x-y) dy - \int_{B(0,1)} \eta(y) u_m(x) dy \\ &= \int_{B(0,1)} \eta(y) [u_m(x-\varepsilon y) - u_m(x)] dy \\ &= \int_{B(0,1)} \eta(y) \int_0^1 \frac{d}{dt} u_m(x-\varepsilon ty) dt dy \\ &= -\varepsilon \int_{B(0,1)} \eta(y) \int_0^1 Du_m(x-\varepsilon ty) \cdot y dt dy. \end{aligned}$$

The first equality follows from the fact that  $\int_{B(0,1)} \eta(y) dy = 1$ , the third equality follows from the Fundamental Theorem of Calculus, and the last follows from the generalized chain rule.

Using the above equality and the fact that  $|y| \leq 1$ ,

$$\int_V |u_m^\varepsilon(x) - u_m(x)| dx \leq \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V |Du_m(x-\varepsilon ty)| dx dt dy.$$

Now let  $z = x - \varepsilon ty$ . Then letting  $V^\varepsilon = \{x \in \mathbb{R}^n \mid x + \varepsilon ty \in V\}$ ,

$$\begin{aligned} &\varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V |Du_m(x-\varepsilon ty)| dx dt dy \\ &= \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_{V^\varepsilon} |Du_m(z)| dz dt dy. \end{aligned}$$

But the support of  $Du_m$  is in  $V$ , so

$$\begin{aligned} \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_{V^\varepsilon} |Du_m(z)| dz dt dy &= \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V |Du_m(z)| dz dt dy \\ &\leq \varepsilon \|Du_m\|_{L^1(V)}. \end{aligned}$$

Thus, when  $u_m$  is smooth,

$$\|u_m^\varepsilon - u_m\|_{L^1(V)} \leq \varepsilon \|Du_m\|_{L^1(V)} \leq C\varepsilon \|Du_m\|_{L^p(V)}$$

by the Corollary to Poincaré's inequality. By the boundedness of  $u_m$ ,

$$u_m^\varepsilon \rightarrow u_m \text{ in } L^1(V) \text{ uniformly in } m, \text{ as } \varepsilon \rightarrow 0. \quad (3)$$

Now, using the Interpolation Inequality,

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \leq \|u_m^\varepsilon - u_m\|_{L^1(V)}^\theta \|u_m^\varepsilon - u_m\|_{L^{p^*}(V)}^{1-\theta}$$

for some  $0 \leq \theta \leq 1$ .

By the Gagliardo-Nirenberg-Sobolev inequality and the boundedness of  $u_m$  in  $W^{1,p}$ ,

$$\|u_m^\varepsilon - u_m\|_{L^{p^*}(V)}^{1-\theta} \leq C \|D(u_m^\varepsilon - u_m)\|_{L^p(V)}^{1-\theta} \leq C'$$

for constants  $C, C'$ .

Therefore, by (3),

$$u_m^\varepsilon \rightarrow u_m \text{ in } L^q(V) \text{ as } \varepsilon \rightarrow 0, \text{ uniformly in } m,$$

where  $u_m \in C^\infty(V) \cap W^{1,p}(V)$ .

Now if  $u_m \in W^{1,q}(V)$ , by Theorem 9, we can find a sequence of functions  $\{v_{m_n}\}_{n=1}^\infty \subseteq C^\infty(V) \cap W^{1,q}(V)$  such that  $v_{m_n} \rightarrow u_m$  in  $W^{1,q}(V)$ . And

$$\begin{aligned} \|v_{m_n}^\varepsilon - u_m^\varepsilon\|_{L^q(V)} &= \int_V \int_{B(0,1)} \eta(y) [v_{m_n}(x - \varepsilon y) - u_m(x - \varepsilon y)] \\ &\leq \|\eta\|_{L^\infty(V)} \int_V \int_{B(0,1)} [v_{m_n}(x - \varepsilon y) - u_m(x - \varepsilon y)] \\ &\leq \|\eta\|_{L^\infty(V)} \|v_{m_n} - u_m\|_{L^q(V)} \end{aligned}$$

Then

$$\begin{aligned} \|v_{m_n}^\varepsilon - v_{m_n}\|_{L^q(V)} &\leq \|v_{m_n}^\varepsilon - u_m^\varepsilon\|_{L^q(V)} + \|u_m^\varepsilon - u_m\|_{L^q(V)} + \|u_m - v_{m_n}\|_{L^q(V)} \\ &\leq \|u_m^\varepsilon - u_m\|_{L^q(V)} + C \|u_m - v_{m_n}\|_{W^{1,q}(V)}, \end{aligned}$$

for some constant  $C$ , and both summands converges to 0 uniformly in  $m$ .

Now we find bounds on  $u_m^\varepsilon(x)$  and  $Du_m^\varepsilon(x)$  and thus establish that for fixed  $\varepsilon > 0$ ,  $\{u_m^\varepsilon(x)\}_{m=1}^\infty$  is uniformly bounded and equicontinuous.

We have

$$\begin{aligned} \|u_m^\varepsilon(x)\| &\leq \int_{B(x,\varepsilon)} |\eta_\varepsilon(x-y)| |u_m(y)| dy \\ &\leq \|\eta_\varepsilon\|_{L^\infty(\mathbf{R}^n)} \|u_m\|_{L^1(V)} \\ &\leq \frac{C_1}{\varepsilon^n}, \end{aligned}$$

for some constant  $C_1$ , so  $\{u_m^\varepsilon(x)\}_{m=1}^\infty$  is uniformly bounded.

$$\begin{aligned} \|Du_m^\varepsilon(x)\| &\leq \int_{B(x,\varepsilon)} |D\eta_\varepsilon(x-y)| |u_m(y)| dy \\ &\leq \|D\eta_\varepsilon\|_{L^\infty(\mathbf{R}^n)} \|u_m\|_{L^1(V)} \\ &\leq \frac{C_2}{\varepsilon^{n+1}}, \end{aligned}$$

for some constant  $C_2$ .

Now

$$\begin{aligned} \|u_m^\varepsilon(x) - u_m^\varepsilon(y)\| &\leq \int_{B(0,\varepsilon)} |\eta_\varepsilon(z)| |u_m(x-z) - u_m(y-z)| dz \\ &\leq \|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\varepsilon)} |u_m(x-z) - u_m(y-z)| dz, \end{aligned}$$

and since  $u_m$  is continuous,  $\{u_m^\varepsilon\}_{m=1}^\infty$  is equicontinuous.

Now we show that there exists a subsequence  $\{u_{m_i}\}_{i=1}^\infty$  of  $\{u_m\}_{m=1}^\infty$  such that

$$\limsup_{j,k \rightarrow \infty} \|u_{m_j} - u_{m_k}\| \leq \delta,$$

for any  $\delta > 0$ .

By the uniform convergence of  $u_m^\varepsilon$  to  $u_m$  in  $L^q(V)$ , we can find  $\varepsilon > 0$  such that  $\|u_m^\varepsilon - u_m\|_{L^q(V)} \leq \frac{\delta}{2}$  for  $m = 1, 2, \dots$

Note that  $\{u_m^\varepsilon\}_{m=1}^\infty$  has support in  $V$ , so by the Arzela-Ascoli Theorem, there exists a subsequence  $\{u_{m_j}^\varepsilon\}_{j=1}^\infty$  converging uniformly in  $V$ . Thus,

$$\limsup_{j,k \rightarrow \infty} \|u_{m_j}^\varepsilon - u_{m_k}^\varepsilon\|_{L^q(V)} = 0.$$

Then

$$\begin{aligned} \limsup_{j,k \rightarrow \infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} &\leq \limsup_{j \rightarrow \infty} \|u_{m_j} - u_{m_j}^\varepsilon\|_{L^q(V)} + \limsup_{j,k \rightarrow \infty} \|u_{m_j}^\varepsilon - u_{m_k}^\varepsilon\|_{L^q(V)} \\ &\quad + \limsup_{k \rightarrow \infty} \|u_{m_k}^\varepsilon - u_{m_k}\|_{L^q(V)} \\ &\leq \frac{\delta}{2} + 0 + \frac{\delta}{2} \\ &= \delta. \end{aligned}$$

Now consider this result with  $\delta = 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots$ . We use a diagonal argument to obtain a subsequence  $\{u_{m_j}\}_{j=1}^\infty$  of  $\{u_m\}_{m=1}^\infty$  such that

$$\limsup_{j,k \rightarrow \infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} = 0$$

as follows:

For  $\delta = 1$ , we obtain a subsequence  $\{u_{m_{1,j}}\}_{j=1}^\infty$  such that

$$\limsup_{j,k \rightarrow \infty} \|u_{m_{1,j}} - u_{m_{1,k}}\|_{L^q(V)} \leq 1.$$

Now for  $\delta = \frac{1}{2}$ , we can obtain a subsequence  $\{u_{m_{2,j}}\}_{j=1}^\infty$  of  $\{u_{1,j}\}_{j=1}^\infty$  such that

$$\limsup_{j,k \rightarrow \infty} \|u_{m_{2,j}} - u_{m_{2,k}}\|_{L^q(V)} \leq \frac{1}{2}.$$

Continuing in a similar manner, for  $\delta = \frac{1}{n}$ , we have a subsequence  $\{u_{m_{n,j}}\}_{j=1}^\infty$  of  $\{u_{m_{n-1,j}}\}_{j=1}^\infty$  such that

$$\limsup_{j,k \rightarrow \infty} \|u_{m_{n,j}} - u_{m_{n,k}}\|_{L^q(V)} \leq \frac{1}{n}.$$

Now form a new subsequence  $\{a_r\}_{r=1}^\infty$  defined by  $a_r = u_{m_{r,r}}$  for  $r = 1, 2, \dots$ . Then for this subsequence,

$$\sup_{j,k \geq n} \|a_j - a_k\|_{L^q(V)} \leq \frac{1}{n}$$

and

$$\limsup_{j,k \rightarrow \infty} \|a_j - a_k\|_{L^q(V)} = 0,$$

the condition that a bounded sequence in  $W^{1,p}(U)$  is precompact in  $L^q(U)$ .  $\square$

#### 4. CONSTRUCTION OF THE INVERSE OF AN ELLIPTIC OPERATOR

We say that  $D^\alpha u$  exists in the weak sense if

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx$$

for all test functions  $\phi \in C_c^\infty(U)$ .

Let  $W^{k,p}(U)$  denote the Sobolev space of locally summable functions  $u : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  such that for each multiindex  $\alpha$ ,  $|\alpha| \leq k$ ,  $D^\alpha u$  exists in the weak sense and  $D^\alpha u \in L^p(U)$ . We can define its norm to be

$$\|u\|_{W^{k,p}(U)} \equiv \left( \sum_{|\alpha| \leq k} \int_U |D^\alpha u|^p dx \right)^{1/p} \quad (1 \leq p < \infty).$$

Also let  $W_0^{k,p}(U)$  be the closure of  $C_c^\infty$  in  $W^{k,p}(U)$  and  $H^k \equiv W^{k,2}$ .

We want to work with the Hilbert space  $H_0^1$  and apply the Lax-Milgram Theorem to make observations about weak solutions of our problem,

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

where

$$L = - \sum_{i,j=1}^n a_{ij}(x) (u_{x_i})_{x_j} + \sum_{i=1}^n b_i(x) u_{x_i} + c(x)u.$$

We will call an operator  $L$  elliptic if there exists a  $\theta > 0$  such that

$$\sum_{i,j=1}^n a_{ij}(x) e_i e_j \geq \theta |e|^2,$$

for a.e.  $x \in U$  and all  $e \in \mathbb{R}^n$ , with  $e_i$  denoting the  $i^{\text{th}}$  entry of  $e$ .

Now we present some motivation for the definition of a weak solution. If  $u$  is smooth, we can take the inner product of both sides of  $Lu = f$  with some function  $v$  and obtain

$$\langle Lu, v \rangle = \langle f, v \rangle.$$

Integrating by parts,

$$\langle Lu, v \rangle = \int_U \left( \sum_{i,j=1}^n a_{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^n b_i(x) u_{x_i} v + c(x)uv \right) dx.$$

Thus, we define  $B : H_0^1(U) \times H_0^1(U) \rightarrow \mathbb{R}$  to be the bilinear form

$$B[u, v] \equiv \int_U \left( \sum_{i,j=1}^n a_{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^n b_i(x) u_{x_i} v + c(x)uv \right) dx. \quad (4)$$

and define a weak solution of this problem to be  $u \in \mathcal{H}$  such that

$$B[u, v] = \langle f, v \rangle_{L^2(U)}.$$

Now we present sufficient conditions for

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

to have a weak solution.

**Theorem 10.** (Lax-Milgram) *Let  $\mathcal{H}$  be a Hilbert space and suppose  $B : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$  is a bilinear form such that*

$$\begin{aligned} (i) \quad & |B[u, v]| \leq \alpha \|u\| \|v\|, \\ (ii) \quad & \beta \|u\|^2 \leq B[u, u], \end{aligned}$$

for constants  $\alpha, \beta > 0$ . Let  $F : \mathcal{H} \rightarrow \mathbb{R}$  be a bounded linear functional. Then there exists a unique  $u \in \mathcal{H}$  such that

$$B[u, v] = F(v)$$

for every  $v \in \mathcal{H}$ .

*Proof.* Fix  $u \in \mathcal{H}$ . Then  $B[u, v]$  is a bounded linear functional with respect to  $v$ . By the Riesz Representation Theorem, there exists a unique  $w \in \mathcal{H}$  such that

$$B[u, v] = \langle w, v \rangle.$$

We define  $A : \mathcal{H} \rightarrow \mathcal{H}$  by  $Au = w$  whenever this inequality holds; that is,

$$B[u, v] = \langle Au, v \rangle.$$

We claim that  $A$  is a bounded linear operator. If  $\lambda_1, \lambda_2 \in \mathbb{R}$  and  $u_1, u_2 \in \mathcal{H}$ ,

$$\begin{aligned} \langle A(\lambda_1 u_1 + \lambda_2 u_2), v \rangle &= B[\lambda_1 u_1 + \lambda_2 u_2, v] \\ &= \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v] \\ &= \lambda_1 \langle Au_1, v \rangle + \lambda_2 \langle Au_2, v \rangle \\ &= \langle \lambda_1 Au_1 + \lambda_2 Au_2, v \rangle \end{aligned}$$

for any  $v \in \mathcal{H}$ , and letting  $v = A(\lambda_1 u_1 + \lambda_2 u_2) - (\lambda_1 Au_1 + \lambda_2 Au_2)$ , we find that  $v = 0$ , so  $A$  is linear.

Now

$$\begin{aligned} \|Au\|^2 &= \langle Au, Au \rangle \\ &= B[u, Au] \\ &\leq \alpha \|u\| \|Au\| \end{aligned}$$

by (i). Thus,

$$\frac{\|Au\|}{\|u\|} \leq \alpha, \tag{5}$$

and hence  $A$  is bounded.

Next we show that  $A$  is one-to-one and the range of  $A$ , denoted  $R(A)$ , is closed in  $\mathcal{H}$ . First,

$$\begin{aligned}\beta\|u\|^2 &\leq B[u, u] \\ &= \langle Au, u \rangle \\ &\leq \|Au\|\|u\|,\end{aligned}$$

so  $\beta\|u\| \leq \|Au\|$ . If  $Au = Av$ ,

$$\beta\|u - v\| \leq \|A(u - v)\| = 0,$$

so  $u = v$  and hence  $A$  is one-to-one. Now if  $\{x_n\}_{n=1}^\infty$  is a sequence in  $\mathcal{H}$  such that  $\{Ax_n\}_{n=1}^\infty$  converges, then  $\{Ax_n\}_{n=1}^\infty$  is a Cauchy sequence. Then

$$\beta\|x_n - x_m\| \leq \|A(x_n - x_m)\| \rightarrow 0,$$

so  $\{x_n\}_{n=1}^\infty$  is a Cauchy sequence as well. Since  $\mathcal{H}$  is a Hilbert space,  $\{x_n\}_{n=1}^\infty$  converges to some  $x \in \mathcal{H}$ . Then

$$\|A(x_n - x)\| \leq \alpha\|x_n - x\| \rightarrow 0$$

by (5), so  $Ax_n \rightarrow Ax$  as  $n \rightarrow \infty$ . Therefore,  $R(A)$  is closed.

Now we show that  $R(A) = \mathcal{H}$ . Suppose  $R(A) \neq \mathcal{H}$ . Let  $P$  be the projection of  $\mathcal{H}$  onto  $R(A)$ . If  $h \in \mathcal{H} \setminus R(A)$ , then  $h - Ph \in R(A)^\perp$  and since  $h \notin R(A)$ ,  $h - Ph \neq 0$ . But then

$$\begin{aligned}\beta\|h - Ph\|^2 &\leq B[h - Ph, h - Ph] \\ &= \langle A(h - Ph), h - Ph \rangle \\ &= 0,\end{aligned}$$

a contradiction. Therefore,  $R(A) = \mathcal{H}$ .

Thus, we can apply the Riesz Representation Theorem again to the functional  $F(v)$  to find a unique  $w \in \mathcal{H}$  such that

$$F(v) = \langle w, v \rangle.$$

Since  $A$  is one-to-one and onto, there exists a unique  $u \in \mathcal{H}$  such that  $w = Au$ . So

$$F(v) = \langle w, v \rangle = \langle Au, v \rangle = B[u, v].$$

□

We now show that there are slightly weaker conditions that our bilinear form from (4) satisfy, and we can use these to obtain a related bilinear form to which the Lax-Milgram Theorem applies.

**Theorem 11.** *For the bilinear form  $B$  defined in (4), there exist constants  $\alpha, \beta > 0$  and  $\gamma \geq 0$  such that*

$$\begin{aligned}(i) &|B[u, v]| \leq \alpha\|u\|_{H_0^1(U)}\|v\|_{H_0^1(U)}, \\ (ii) &\beta\|u\|_{H_0^1(U)}^2 \leq B[u, u] + \gamma\|u\|_{L^2(U)}^2.\end{aligned}$$

*Proof.* We have

$$\begin{aligned}
B[u, v] &\leq \|a_{ij}(x)\|_{L^\infty(U)} \int_U \sum_{i,j=1}^n |u_{x_i}| |v_{x_j}| dx \\
&\quad + \|b_i(x)\|_{L^\infty(U)} \int_U \sum_{i=1}^n |u_{x_i}| |v| dx + \|c(x)\|_{L^\infty(U)} \int_U |u| |v| dx \\
&\leq \|a_{ij}(x)\|_{L^\infty(U)} n \int_U |Du| |Dv| dx \\
&\quad + \|b_i(x)\|_{L^\infty(U)} n^{1/2} \int_U |Du| |v| dx + \|c(x)\|_{L^\infty(U)} \int_U |u| |v| dx,
\end{aligned}$$

by Minkowski's inequality. But  $\int_U |Dw| dx \leq \sum_{|\alpha|=1} \int_U |D^\alpha w| dx$  for  $w \in H_0^1$ , so

$$\begin{aligned}
B[u, v] &\leq \|a_{ij}(x)\|_{L^\infty(U)} n \int_U |Du| dx \int_U |Dv| dx \\
&\quad + \|b_i(x)\|_{L^\infty(U)} n^{1/2} \int_U |Du| |v| dx + \|c(x)\|_{L^\infty(U)} \int_U |u| |v| dx, \\
&\leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}
\end{aligned}$$

for an appropriate constant  $\alpha > 0$ .

Now since  $L$  is an elliptic operator,

$$\begin{aligned}
\int_U \theta |Du|^2 dx &\leq \int_U \sum_{i,j=1}^n a_{ij}(x) u_{x_i} u_{x_j} dx \\
&= B[u, u] - \int_U \left( \sum_{i=1}^n b_i(x) u_{x_i} u + c(x) u^2 \right) dx \\
&\leq B[u, u] + \|b_i(x)\|_{L^\infty(U)} n^{1/2} \int_U |Du| |u| dx + \|c(x)\|_{L^\infty(U)} \int_U u^2 dx.
\end{aligned}$$

But by Cauchy's inequality with  $\varepsilon$ , we have

$$\int_U |Du| |u| dx \leq \int_U \left[ \varepsilon |Du|^2 + \frac{1}{4\varepsilon} |u|^2 \right] dx.$$

We will choose  $\varepsilon$  sufficiently small to ensure that

$$\varepsilon \|b_i(x)\|_{L^\infty(U)} n^{1/2} < \frac{\theta}{2}.$$

Then

$$\frac{\theta}{2} \int_U |Du|^2 dx \leq B[u, u] + C_1 \|u\|_{L^2(U)}^2, \tag{6}$$

for some constant  $C_1$ .

Poincaré's inequality states that

$$\|u\|_{L^2(U)} \leq C_2 \|Du\|_{L^2(U)}$$

for some constant  $C_2$ . But

$$\begin{aligned}\|u\|_{H_0^1(U)}^2 &= \sum_{|\alpha| \leq 1} \int_U |D^\alpha u|^2 dx \\ &= \|u\|_{L^2(U)}^2 + \int_U \sum_{i=1}^n u_{x_i}^2 dx \\ &= \|u\|_{L^2(U)}^2 + \|Du\|_{L^2(U)}^2.\end{aligned}$$

By (6), we have

$$\begin{aligned}\frac{\theta}{2}\|u\|_{H_0^1(U)}^2 &\leq \frac{\theta}{2}\|u\|_{L^2(U)}^2 + B[u, u] + C_1\|u\|_{L^2(U)}^2 \\ &\leq B[u, u] + \left(\frac{\theta}{2} + C_1\right)\|u\|_{L^2(U)}^2\end{aligned}$$

so

$$\beta\|u\|_{H_0^1(U)} \leq B[u, u] + \gamma\|u\|_{L^2(U)}$$

for some  $\beta > 0$  and  $\gamma \geq 0$ . □

**Theorem 12.** *There exists a constant  $\gamma \geq 0$  such that for each  $\mu \geq \gamma$  and each  $f \in L^2(U)$ , there is a unique weak solution  $u \in H_0^1(U)$  of*

$$\begin{cases} Lu + \mu u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

*Proof.* Let  $\alpha, \beta, \gamma$  be as in Theorem 11. Then we define

$$B_\mu[u, v] \equiv B[u, v] + \mu\langle u, v \rangle.$$

Since  $B[u, v]$  is a bounded linear functional with respect to  $v$ , so is  $B_\mu[u, v]$ . Now

$$\begin{aligned}B_\mu[u, v] &= B[u, v] + \mu\langle u, v \rangle \\ &\leq \alpha\|u\|_{H_0^1(U)}\|v\|_{H_0^1(U)} + \mu\|u\|_{L^2(U)}\|v\|_{L^2(U)} \\ &\leq (\alpha + \mu)\|u\|_{H_0^1(U)}\|v\|_{H_0^1(U)},\end{aligned}$$

so  $B_\mu[u, v]$  satisfies condition (i) of the Lax-Milgram Theorem.

Using condition (ii) of Theorem 11,

$$\begin{aligned}B_\mu[u, u] &= B[u, u] + \mu\langle u, u \rangle \\ &\geq \beta\|u\|_{H_0^1(U)}^2 - \gamma\|u\|_{L^2(U)}^2 + \mu\|u\|_{L^2(U)}^2 \\ &\geq \beta\|u\|_{H_0^1(U)}^2,\end{aligned}$$

so we obtain condition (ii) of the Lax-Milgram Theorem. Then for  $f \in L^2(U)$ , define

$$F(v) \equiv \langle f, v \rangle_{L^2(U)}.$$

Now since inner products are linear and  $F(v) \equiv \langle f, v \rangle \leq \|f\|\|v\|$ ,  $F(v)$  is a bounded linear functional.

Thus, we apply the Lax-Milgram Theorem to obtain a unique  $u \in H_0^1(U)$  such that  $B_\mu[u, v] = F(v) = \langle f, v \rangle_{L^2(U)}$ . The bilinear form  $B_\mu$  corresponds to the

operator  $L_\mu \equiv L + \mu I$  (where  $I$  is the identity), so we have a unique weak solution  $u \in H_0^1(U)$  of

$$\begin{cases} Lu + \mu u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

□

**Theorem 13.** *Either there exists a unique weak solution  $u \in H_0^1(U)$  such that*

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \quad (7)$$

for each  $f \in L^2(U)$  or else there exists a nonzero weak solution to the homogeneous problem

$$\begin{cases} Lu = 0 & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \quad (8)$$

*Proof.* Let  $\gamma, \mu$  be as in Theorem 12 and choose  $\mu = \gamma$ . Then we can define the bilinear form

$$B_\gamma[u, v] = B[u, v] + \gamma \langle u, v \rangle$$

satisfying the conditions of the Lax-Milgram Theorem. Then for any  $g \in L^2(U)$ , there exists a unique  $u \in H_0^1(U)$  such that

$$B_\gamma[u, v] = \langle g, v \rangle. \quad (9)$$

We will write

$$u = L_\gamma^{-1}g$$

whenever (9) holds. For  $c_1, c_2 \in \mathbb{R}$  and  $g_1, g_2 \in L^2(U)$

$$\begin{aligned} B_\gamma[L_\gamma^{-1}(c_1g_1 + c_2g_2), v] &= \langle c_1g_1 + c_2g_2, v \rangle \\ &= c_1 \langle g_1, v \rangle + c_2 \langle g_2, v \rangle \\ &= B[c_1L_\gamma^{-1}g_1, v] + B[c_2L_\gamma^{-1}g_2, v] \\ &= B[c_1L_\gamma^{-1}g_1 + c_2L_\gamma^{-1}g_2, v] \end{aligned}$$

for all  $v \in H_0^1(U)$ . Then  $L_\gamma^{-1}$  is linear.

Now (7) is equivalent to

$$\begin{cases} Lu + \gamma u = f + \gamma u & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases}$$

so (7) holds if and only if

$$B_\gamma[u, v] = \langle \gamma u + f, v \rangle,$$

which we can write as

$$\begin{aligned} u &= L_\gamma^{-1}(\gamma u + f) \\ &= \gamma L_\gamma^{-1}u + L_\gamma^{-1}f \\ &= Ku + h \end{aligned}$$

for  $K = \gamma L_\gamma^{-1}$  and  $h = L_\gamma^{-1}f$ . Then (7) has a unique solution for each  $f \in L^2(U)$  if and only if  $u - Ku = h$  has a unique solution for each  $h \in L^2(U)$ .

We will show that  $K$  is a compact operator. First, we show that  $K$  is linear and bounded. We have

$$K(c_1g_1 + c_2g_2) = c_1\gamma L_\gamma^{-1}g_1 + c_2\gamma L_\gamma^{-1}g_2 = c_1Kg_1 + c_2Kg_2,$$

so  $K$  is linear. And

$$\begin{aligned} \beta \|u\|_{H_0^1(U)}^2 &\leq B_\gamma[u, u] \\ &= \langle g, u \rangle \\ &\leq \|g\|_{L^2(U)} \|u\|_{L^2(U)} \\ &\leq \|g\|_{L^2(U)} \|u\|_{H_0^1(U)}, \end{aligned}$$

Then we have  $\beta \|u\|_{H_0^1(U)} \leq \|g\|_{L^2(U)}$ . But

$$\begin{aligned} \|Kg\|_{H_0^1(U)} &= \|\gamma L_\gamma^{-1}g\|_{H_0^1(U)} \\ &= \frac{\gamma}{\beta} \beta \|u\|_{H_0^1(U)} \\ &= \frac{\gamma}{\beta} \|g\|_{L^2(U)} \\ &\leq \frac{\gamma}{\beta} \|g\|_{H_0^1(U)}, \end{aligned}$$

so  $K$  is bounded. Now by the Rellich-Kondrachov Theorem,  $H_0^1(U) \subset\subset L^2(U)$ , so  $K$  is compact.

Thus, the Fredholm Alternative implies the following result: either the equation

$$u - Ku = h \tag{10}$$

has a unique solution  $u \in L^2(U)$  for each  $h \in L^2(U)$  or the equation

$$u - Ku = 0 \tag{11}$$

has a nonzero solution  $u \in L^2(U)$ .

We have already established in our derivation of  $K$  that (10) is equivalent to (7). It remains to show that (11) is equivalent to (8). If we have a solution  $u$  to (8),

$$u = Ku = \gamma L_\gamma^{-1}u,$$

so that

$$B_\gamma[u, v] = B_\gamma[\gamma L_\gamma^{-1}u, v] = \langle \gamma u, v \rangle$$

for all  $v \in H_0^1(U)$ , and  $B[u, v] = 0$ , so we have that  $u$  solves (8). And if we have a solution  $u$  to (8),

$$B_\gamma[u, v] = \langle \gamma u, v \rangle,$$

and hence

$$u = \gamma L_\gamma^{-1}u = Ku.$$

□

**Theorem 14.** *There exists an at most countable set  $\Sigma$  of real numbers such that*

$$\begin{cases} Lu = \lambda u + f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \quad (12)$$

*has a unique weak solution for  $f \in L^2(U)$  if and only if  $\lambda \notin \Sigma$ . Furthermore, if  $\Sigma$  is an infinite set, then  $\Sigma = \{\lambda_k\}_{k=1}^{\infty}$ , with  $\lambda_k \rightarrow \infty$  as  $k \rightarrow \infty$ .*

*Proof.* Let  $\gamma$  be as in Theorem 11. Applying the operator  $L - \lambda I$  to Theorem 13, there exists a unique solution to (12) if and only if  $u \equiv 0$  is the only solution to

$$\begin{cases} Lu = \lambda u & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

This system is equivalent to

$$\begin{cases} L_\gamma u = (\lambda + \gamma)u & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases} \quad (13)$$

using the notation of  $L_\gamma \equiv L + \gamma I$  as in Theorem 13. Now from the proof of Theorem 13, (13) has a weak solution if and only if

$$u = (\lambda + \gamma)L_\gamma^{-1}u = \frac{\lambda + \gamma}{\gamma}Ku,$$

where we have made the substitution  $K = \gamma L_\gamma^{-1}$ . Thus, for  $\lambda \neq -\gamma$ , there exists a unique weak solution to (12) if and only if  $\frac{\gamma}{\lambda + \gamma}$  is not an eigenvalue of  $K$ .

As before,  $K$  is a compact operator. By the Spectral Theorem, the set of eigenvalues of  $K$  is a countable sequence tending to 0. Then there is a unique solution to (10) for all  $\lambda$  except for a countable sequence  $\{\lambda_k\}_{k=1}^{\infty}$ . But  $\lambda_k \rightarrow \infty$  if and only if  $\frac{\gamma}{\lambda_k + \gamma} \rightarrow 0$ . □

Now consider the case of  $f \equiv 0$  in Theorem 14. Then we have the system

$$\begin{cases} Lu = \lambda u & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

Since  $u \equiv 0$  is a solution to this system, if  $\lambda \notin \Sigma$ , then  $u \equiv 0$  is the unique solution, so  $\lambda$  cannot be an eigenvalue. But if  $\lambda \in \Sigma$ , this system must have a solution besides  $u \equiv 0$ , and this solution is an eigenvector of  $L$ . Therefore,  $\Sigma$  is the set of all eigenvalues of  $L$ .

## 5. NON-ISOMETRIC ISOSPECTRAL DOMAINS

Now we return to our main problem, the question of whether one can hear the shape of a drum. One can hear a sound that is some mixture of all the frequencies obtained from the sequence of eigenvalues of the laplacian. The tone with the lowest frequency is called the fundamental frequency and all of the other tones are known as overtones. When a human hears a sound, the intensity or amplitude of the wave may be different, but the shape of the wave will be the same (up to scale) for any given drum. A person with perfect pitch can thus in theory “hear” all the eigenvalues. Our question is whether knowing all of these eigenvalues enables one to recreate the drum, or domain, that produced them.

We will now present some examples of isospectral domains that are not isometric. Without loss of generality, we can let

$$\lambda_1 \leq \lambda_2 \leq \dots$$

Now we turn to the questions that we can answer about the domain  $U$ . We will consider the end behavior of the sequence of eigenvalues.

By a conjecture of Lorenz later proved by Weyl, we have the formula

$$N(\lambda) \equiv \sum_{\lambda_n < \lambda} 1 \sim \frac{|U|}{2\pi} \lambda.$$

This formula states that the number of eigenvalues less than  $\lambda$ ,  $N(\lambda)$ , is about  $\frac{|U|}{2\pi} \lambda$ , where  $|U|$  is the area of the domain  $U$ . More precisely, this means that

$$\lim_{n \rightarrow \infty} \frac{N(\lambda)}{\lambda} = \frac{|U|}{2\pi}.$$

Now suppose  $U_1$  and  $U_2$  are two domains that have different area. Then  $\frac{|U_1|}{2\pi} \neq \frac{|U_2|}{2\pi}$ , so

$$\lim_{n \rightarrow \infty} \frac{N_{U_1}(\lambda)}{\lambda} \neq \lim_{n \rightarrow \infty} \frac{N_{U_2}(\lambda)}{\lambda}.$$

But then  $N_{U_1}(\lambda) \neq N_{U_2}(\lambda)$  for some  $\lambda$ . This shows that  $U_1$  and  $U_2$  cannot be isospectral. Furthermore, if we know the spectrum of  $U$ , we can use this formula to calculate

$$\lim_{n \rightarrow \infty} \frac{N(\lambda)}{\lambda},$$

which gives us  $|U|$ .

Now by a theorem of Pleijel [2],

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|U|}{2\pi t} - \frac{L}{4\sqrt{2\pi t}} \text{ as } t \rightarrow 0,$$

where  $L$  is the length of  $\partial U$ . If  $U_1$  and  $U_2$  are domains with the same area but different boundary length, then

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \neq \sum_{n=1}^{\infty} e^{-\mu_n t}$$

for some  $t$ , where  $\{\lambda_n\}_{n=1}^{\infty}$  is the sequence of eigenvalues for  $U_1$  and  $\{\mu_n\}_{n=1}^{\infty}$  is the sequence of eigenvalues for  $U_2$ . But this implies that  $\lambda_i \neq \mu_i$  for some  $i$ . Thus,  $U_1$  and  $U_2$  are not isospectral. Furthermore, if we know the spectrum of a domain  $U$ ,

then we can calculate  $\sum_{n=1}^{\infty} e^{-\lambda_n t}$ , which gives us  $\frac{|U|}{2\pi t} - \frac{L}{4\sqrt{2\pi t}}$ . But we know  $|U|$  already, so this gives us  $L$  as well.

Note in particular that if the domain  $U$  is circular, we will be able to "hear" this. By the isoperimetric inequality,

$$4\pi|U| \leq L^2$$

with equality only in the case of the circle. This implies that we will have a circular drum if and only if

$$\frac{|U|}{2\pi t} - \frac{L}{4\sqrt{2\pi t}} = \frac{L^2}{8\pi^2 t} - \frac{L}{4\sqrt{2\pi t}},$$

and we can calculate both  $|U|$  and  $L$  as above.

According to a conjecture of Kac, we have this formula for the spectrum of a smooth domain  $U$  with smooth holes:  $\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|U|}{2\pi t} - \frac{L}{4\sqrt{2\pi t}} + \frac{1}{6}(1-r)$  as  $t \rightarrow 0$ , where  $r$  is the number of smooth holes. Thus, we expect to be able to "hear" the number of holes as well.

Though these results tell us a great deal about the domain  $U$ , it turns out that there exist isospectral domains that are not isometric. In 1964, Milnor showed that there exist two 16-dimensional tori that are isospectral but not isometric. As a physical example of drums that sound the same, this result was not necessarily so enlightening. Nevertheless, this and other such examples through the years led many to conjecture that even in lower dimensional cases, there are simple examples of non-isometric isospectral domains.

Finally, in 1992, Gordon et. al., exhibited a counterexample for planar domains [3]. Buser et. al. [4], present particularly simple cases, which we will discuss in this section. The general strategy will be to construct domains which can be subdivided into congruent subdomains. By putting these subdomains together in two different ways, we can obtain non-isometric isospectral domains.

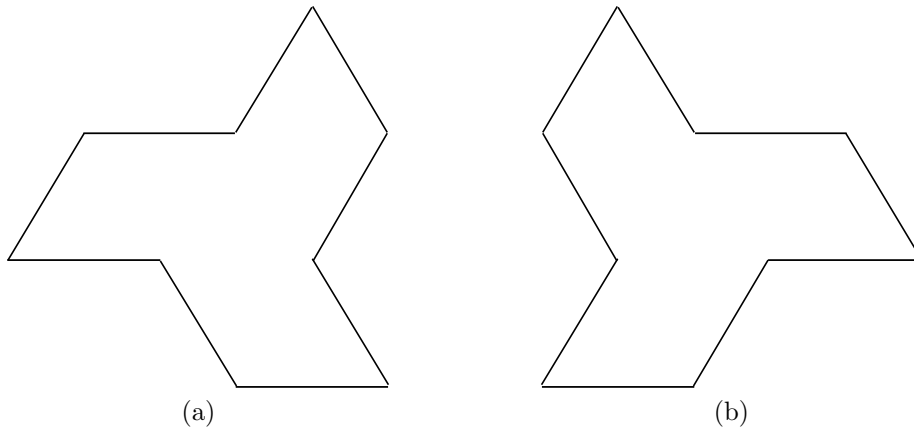


FIGURE 1. A simple example from Buser et. al. [4].

The two figures are not isometric. First, we divide the figure 1a and figure 1b into congruent triangular subdomains as shown. Now we use dots, dashes, and solid lines for each side of the central triangle in both figures. And we use dots, dashes,

and solid lines for the sides of the other triangles so that if two triangles share a side, they are reflections of one another.

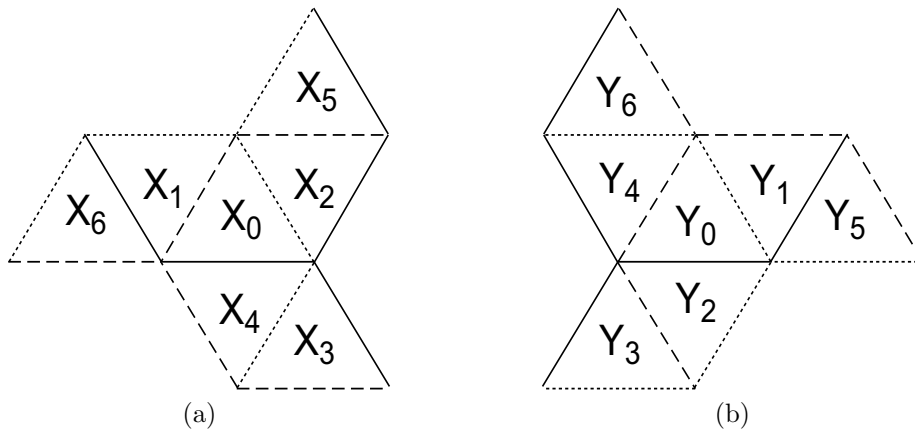


FIGURE 2. Subdividing the two domains into congruent triangles.

By a suitable transplantation mapping, we can write eigenfunctions on subdomains of figure 2b in terms of eigenfunctions on subdomains of figure 2a. Let  $X$  be an eigenfunction of the laplacian of the domain in figure 1a. Then we define  $X_0, X_1, \dots, X_6$  to be the restriction of  $X$  to subdomain  $i$ ,  $i = 0, 1, \dots, 6$  as shown. We will attempt to represent eigenfunctions  $Y_i, i = 0, 1, \dots, 6$  in terms of  $X_0, X_1, \dots$ , and  $X_6$ . Also, we represent the isometry from triangle  $X_j$  on the right to triangle  $Y_i$  on the left by  $\tau_{ij}$ .

Now let us consider an example of transplantation. Suppose  $Y_0 = X_1(\tau_{10}) +$  other terms. From figure 2a, in order to have the boundary term of  $X_1$  cancel over dashes, we must have  $X_0$  as an eigenfunction in the triangle in the reflection over dashes. Then  $X_0(\tau_{04})$  must be in triangle  $Y_4$  of figure 2b.

Now what happens if, in figure 2a, the side of a triangle intersects  $\partial U$ , as is the case for reflection over dots in triangle  $X_1$ ? For the Dirichlet boundary condition,  $X = 0, x \in \partial U$ , we can use odd reflection to show that the necessary eigenfunction in triangle  $Y_1$  of figure 2b is  $-X_1 \circ \sigma$ , where  $\sigma$  is reflection over the boundary.

Henceforth, we will drop  $\tau$  and  $\sigma$  from our notation, with the understanding that an eigenfunction for a subdomain in figure 1b is actually a composition of an eigenfunction of a subdomain in figure 1a with some  $\tau$  or  $\sigma$ .

We will now construct a correspondence between figures 2a and 2b so that the eigenfunction on the right will have continuous partial derivatives and satisfy the Dirichlet boundary condition. We set  $Y_0 = X_1 + X_2 + X_4$  and use the method of reflection above to find  $Y_1, \dots, Y_6$ . Since, in figure 2b, triangle  $Y_1$  is the reflection over dots of triangle  $Y_0$ , we must find the reflection of eigenfunctions  $X_1, X_2$ , and  $X_4$  over dots in figure 1a. These are, respectively,  $-X_1, X_0$ , and  $X_3$ .

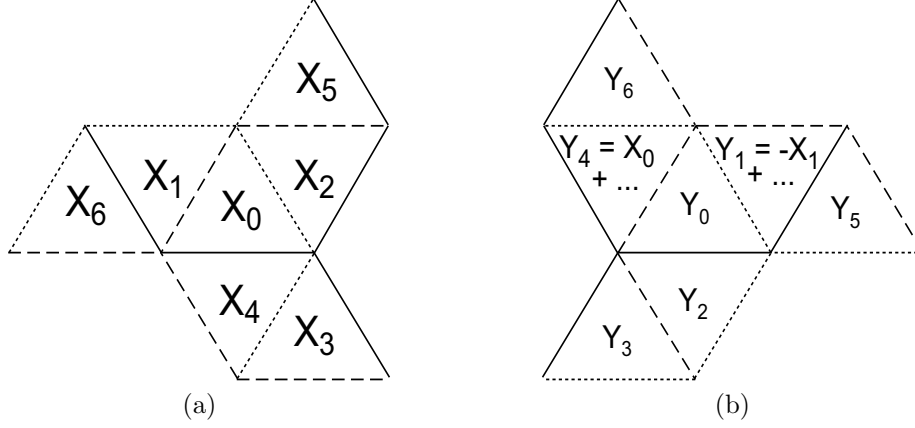


FIGURE 3. Constructing the isometry.

Thus,  $Y_1 = -X_1 + X_0 + X_3$ . Similarly, we have

$$\begin{aligned} Y_2 &= X_6 - X_2 + X_0 \\ Y_3 &= -X_6 - X_5 + X_1 \\ Y_4 &= X_0 + X_5 - X_4 \\ Y_5 &= -X_6 + X_4 - X_3 \\ Y_6 &= X_2 - X_5 - X_3. \end{aligned}$$

Thus, we have a map for each  $\lambda$  taking the  $\lambda$ -eigenspace for figure 1a to the corresponding  $\lambda$ -eigenspace for figure 1b. This correspondence holds for each  $\lambda$ , so the domains in the two figures are isospectral.

The method of transplantation to create a correspondence between two domains will work for many different planar figures. Buser et. al. uses a theorem of Sunada to construct some more [4]. We can use the method of transplantation even if there are holes in the subdivided triangles that we used for our example or if we replace the equilateral triangle that we have used in our example by another polygon. For example, in figure 4, the correspondences are exactly the same as in our example.

We conclude with an example from [5] in which we can calculate the eigenvalues explicitly.

Note that both domains are the union of triangles and rectangles, for which the eigenvalues are well-known. In 6a, 6b, 7a, and 7b, these eigenvalues are

$$\begin{aligned} 6a: \lambda &= (m^2 + n^2)\pi^2, m, n \in \mathbb{Z}; \\ 6b: \lambda &= \left[ \left(\frac{i}{2}\right)^2 + \left(\frac{j}{2}\right)^2 \right] \pi^2, i, j \in \mathbb{Z}, i > j; \\ 7a: \lambda &= \left[ \left(\frac{M}{2}\right)^2 + N^2 \right] \pi^2, M, N \in \mathbb{Z}; \\ 7b: \lambda &= \frac{1}{2}(I^2 + J^2)\pi^2, i, j \in \mathbb{Z}, I > J. \end{aligned}$$

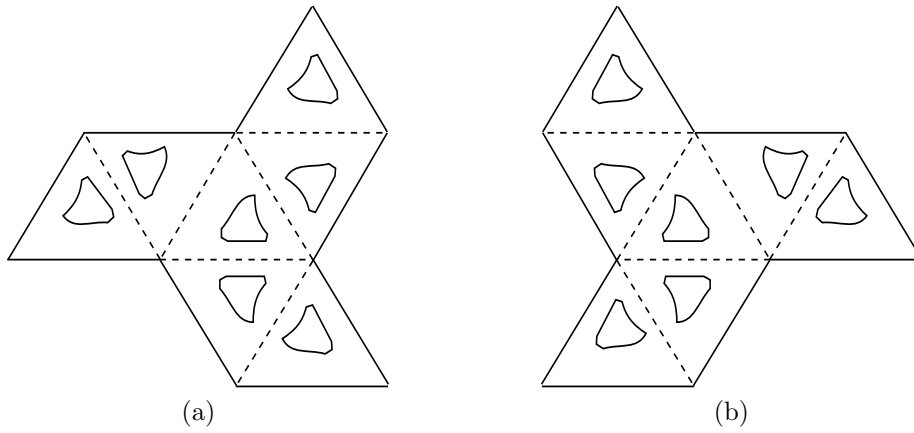


FIGURE 4. Even for shapes with complicated holes, we can still apply the same isometry.

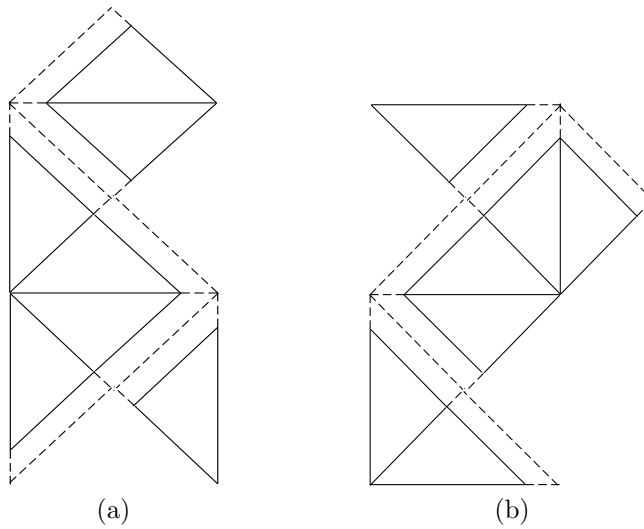


FIGURE 5. Isospectral domains where eigenvalues can be calculated exactly.

Now we will show that an eigenvalue on domain 5a is an eigenvalue on domain 5b. First we note that the small triangles are isometric, so we do not need to consider their spectra. Now if  $\lambda = (m^2 + n^2)\pi^2$ , let  $M = 2m$  and  $N = n$ . Then  $\lambda$  is an eigenvalue of the domain of 5b as well. Now if  $\lambda = ((\frac{i}{2})^2 + (\frac{j}{2})^2)\pi^2$ ,  $i > j$  and  $i$  is odd,  $j$  is even, let  $M = i$ ,  $N = \frac{j}{2}$  while if  $i$  is even and  $j$  is odd, let  $M = \frac{i}{2}$  and  $N = j$ . And if  $\lambda = ((\frac{i}{2})^2 + (\frac{j}{2})^2)\pi^2$ ,  $i > j$  with both  $i, j$  odd or both  $i, j$  even, let  $I = \frac{i-j}{2}$ ,  $J = \frac{i+j}{2}$ . Then  $\lambda$  is again an eigenvalue of domain 5b.

And we show that an eigenvalue on domain 5b is an eigenvalue on domain 5a. For an eigenvalue  $\lambda = ((\frac{M}{2})^2 + N^2)\pi^2$ , if  $M$  is even, let  $m = \frac{M}{2}$  and let  $n = N$ . If  $M$  is odd, let  $i = \max(M, 2N)$  and  $j = \min(M, 2N)$ . Then  $\lambda$  is an eigenvalue of

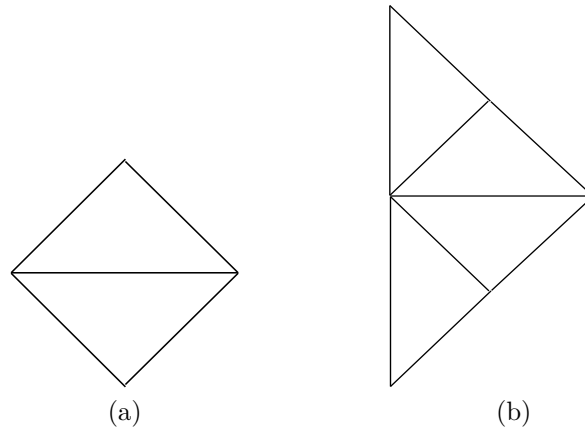


FIGURE 6. Subdivisions of the domain of figure 5a.

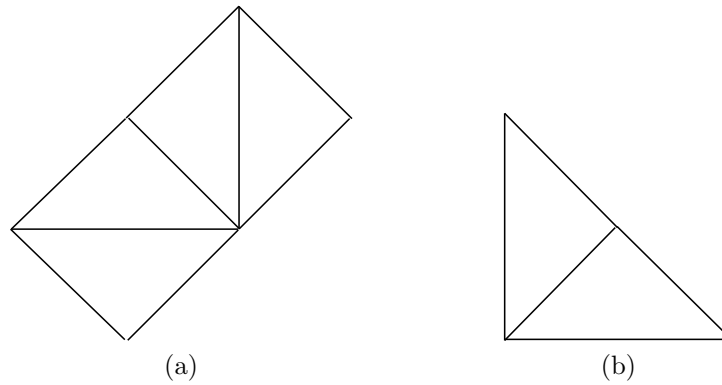


FIGURE 7. Subdivisions of the domain of figure 5b.

domain 5a. Now suppose  $\lambda = \frac{1}{2}(I^2 + J^2)\pi^2, I > J$ . Let  $i = I + J$  and  $j = I - J$ . Then  $\lambda$  is an eigenvalue of domain 5a as well.

## 6. ACKNOWLEDGEMENTS

I would like to thank Professor Richard Schoen for his guidance on this honors thesis and for his infinite patience in explaining important points, for which I am much more grateful than I can express in this short note.

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