

Ph.D. Qualifying Exam, Real Analysis
Spring 2009, part I

Do all five problems. Write your solution for each problem in a separate blue book.

- 1** Fix a real number $p \in (1, \infty)$, and define a linear operator $T : L^p(0, \infty) \rightarrow L^p(0, \infty)$ by $(Tf)(x) = \frac{1}{x} \int_0^x f(y) dy$.

a. Show that

$$\|Tf\|_{L^p(0, \infty)} \leq \frac{p}{p-1} \|f\|_{L^p(0, \infty)}.$$

for all $f \in L^p(0, \infty)$. (Hint: write $|f(y)| = y^{-\frac{p-1}{p^2}} [y^{\frac{p-1}{p^2}} |f(y)|]$ and apply Hölder's inequality.)

b. Show that the operator norm of T is equal to $\frac{p}{p-1}$.

- 2** Suppose that X, Y are Banach spaces, and let \mathcal{T}_s denote the norm topology on X , \mathcal{U}_s the norm topology on Y . Let \mathcal{T}_w denote the weak topology on X , and \mathcal{U}_w denote the weak topology on Y .

a. Show that (X, \mathcal{T}_w) is (T3), i.e. if $x \in X$ then $\{x\}$ is \mathcal{T}_w -closed, and if $x \in X$, and $C \subset X$ is \mathcal{T}_w -closed, $x \notin C$, then there exist \mathcal{T}_w -open sets U, V such that $U \cap V = \emptyset$, $x \in U$, $C \subset V$.

b. Show that a linear map $T : X \rightarrow Y$ is continuous as a map from (X, \mathcal{T}_s) to (Y, \mathcal{U}_s) if and only if it is continuous as a map from (X, \mathcal{T}_w) to (Y, \mathcal{U}_w) .

- 3** Suppose that $\{g_n\}$ is a sequence of positive continuous functions on $[0, 1]$, μ is a positive Borel measure on $[0, 1]$ and that (i) $\lim_{n \rightarrow \infty} g_n(x) = 0$ a.e., (ii) $\int_0^1 g_n dx = 1$ for all n and

$$(iii) \quad \lim_{n \rightarrow \infty} \int_0^1 f g_n dx = \int_0^1 f d\mu$$

for every continuous function $f \in C[0, 1]$. Does it follow that μ is mutually singular with respect to the Lebesgue measure? Prove this or give a counterexample (and prove that it is a counterexample).

- 4** Let $s(\mathbb{Z})$ denote the vector space of rapidly decreasing bi-infinite complex-valued sequences, i.e. sequences $a = \{a_n\}_{n \in \mathbb{Z}} : \mathbb{Z} \rightarrow \mathbb{C}$ such that $\rho_k(\{a_n\}) = \sup |n^k a_n| < \infty$ for all $k \geq 0$ integers, and let \mathcal{T} be the weakest topology on $s(\mathbb{Z})$ in which the functions

$$f_{k,b}(a) = \rho_k(a - b) : s(\mathbb{Z}) \rightarrow [0, \infty),$$

$k \geq 0, b \in s(\mathbb{Z})$, are continuous.

a. Show that \mathcal{T} is metrizable, and give an explicit metric giving rise to the topology \mathcal{T} .

b. Show that there exists no norm $\|\cdot\|$ on $s(\mathbb{Z})$ such that \mathcal{T} is the topology given by the norm $\|\cdot\|$.

- 5** A Banach space B is *uniformly convex* if for every $\epsilon \in (0, 1)$ there exists $\eta < 1$ such that if $x, y \in B$, $\|x\| = \|y\| = 1$ and $\|x - y\| > 2\epsilon$ then $\|\frac{1}{2}(x + y)\| < \eta$.

a. Show that every Hilbert space is uniformly convex, and one may take $\eta = (1 - \epsilon^2)^{1/2}$.

b. For $1 < p < \infty$, show that L^p is uniformly convex. (Hint: For (b), write $\int |f + g|^p d\mu = \int |\phi + \psi|^p d\nu$ where $\phi = \frac{f}{(|f|^p + |g|^p)^{1/p}}$, $\psi = \frac{g}{(|f|^p + |g|^p)^{1/p}}$ and $\nu = (|f|^p + |g|^p) \mu$. This reduces the general case to that where the functions are bounded by 1 and the measure space has total mass 2. Use the convexity of the function x^p in $[-1, 1]$.)

Ph.D. Qualifying Exam, Real Analysis
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Do all five problems. Write your solution for each problem in a separate blue book.

- 1 Two quick problems.
 - a. Suppose that $f \in C(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. Let M_f be the multiplication operator on $L^p(\mathbb{T})$, $p \in [1, \infty)$, $(M_f\phi)(x) = f(x)\phi(x)$ for $\phi \in L^p(\mathbb{T})$. Show that the spectrum of M_f is $\sigma(M_f) = f(\mathbb{T})$.
 - b. Suppose that X is a Banach space, D is a dense subspace, $A_n \in \mathcal{L}(X)$, $n \in \mathbb{N}$, $A \in \mathcal{L}(X)$. Show that $A_n \rightarrow A$ in the strong operator topology if and only if $\{\|A_n\|_{\mathcal{L}(X)} : n \in \mathbb{N}\}$ is bounded and $A_n x \rightarrow Ax$ (in the topology of X) for all $x \in D$.
- 2
 - a. Suppose that Y is a normed complex vector space with norm $\|\cdot\|$, and $\lambda : Y \rightarrow \mathbb{C}$ is linear but is not continuous. Show that $N = \lambda^{-1}(\{0\})$ is dense in Y .
 - b. Show that $L^2([0, 1])$ has an orthonormal basis $\{\phi_n : n \in \mathbb{N}\}$ such that each $\phi_n \in C^1([0, 1])$ and $\phi'_n(1/4) = \phi_n(1/2)$ for each n .
- 3 Suppose that $K \in L^p([0, 1] \times [0, 1])$, $1 < p < \infty$. Let q be the dual exponent, $p^{-1} + q^{-1} = 1$.
 - a. For $f \in L^q([0, 1])$, let $(Af)(x) = \int K(x, y)f(y)dy$. Show that $(Af)(x)$ indeed exists for almost every x and $A \in \mathcal{L}(L^q([0, 1]), L^p([0, 1]))$.
 - b. Suppose that for every $f \in L^q([0, 1])$, $(Af)(x) = 0$ for almost every x . Show that $K = 0$ a.e.
- 4 Let $S'(\mathbb{R}^n)$ denote the set of tempered distributions, and $H^s(\mathbb{R}^n)$ be the subspace of $S'(\mathbb{R}^n)$ consisting of distributions f whose Fourier transforms are functions \hat{f} such that $\|f\|_{H^s(\mathbb{R}^n)}^2 = \int (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi < +\infty$.
 - a. Show that if $2 \leq p \leq \infty$ then for $s > \frac{n}{2} - \frac{n}{p}$ every element of $H^s(\mathbb{R}^n)$ is in $L^p(\mathbb{R}^n)$, and there exists $C = C(n, p, s)$ such that $\|f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}$ for every $f \in H^s(\mathbb{R}^n)$. (Hint: consider $p = 2, \infty$ first.)
 - b. For $p = \infty$, show that this is sharp, namely if $s = \frac{n}{2}$, there is no constant C such that $\|f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}$ for every $f \in H^s(\mathbb{R}^n)$.
- 5 Suppose that X is a Banach space over \mathbb{C} , M and N are closed subspaces of X , and let $M + N = \{x \in X : \exists m \in M, n \in N, x = m + n\}$.
 - a. Show that $M + N$ is closed if and only if there exists $C > 0$ such that for all $x \in M + N$ there exist $m \in M, n \in N$ such that $x = m + n$ and $\|m\| + \|n\| \leq C\|x\|$.
 - b. Suppose that $\ell_M : M \rightarrow \mathbb{C}$ and $\ell_N : N \rightarrow \mathbb{C}$ are continuous linear functionals and $\ell_M|_{M \cap N} = \ell_N|_{M \cap N}$. Show that if $M + N$ is closed, then there exists $\ell \in X^*$ such that $\ell|_M = \ell_M$ and $\ell|_N = \ell_N$.
 - c. Give an example of a Banach space X and closed subspaces M, N such that $M \cap N = \{0\}$ but $M + N$ is *not* closed.