**Ph.D. Qualifying Exam, Real Analysis**

**Spring 2007, part I**

Do all the problems. Write your solution for each problem in a separate blue book.

**1** Two short problems:

**a.** Suppose that $X$ and $Y$ are Banach spaces, and $A_n$, $n = 1, 2, 3, \ldots$, and $A$ are bounded linear operators from $X$ to $Y$. Suppose also that $A_n \to A$ in the weak operator topology, i.e. the weakest topology on bounded linear operators $\mathcal{L}(X, Y)$ in which the maps

$$E_{\ell,x} : \mathcal{L}(X, Y) \ni A \mapsto \ell(Ax),\ x \in X,\ \ell \in Y^*,$$

are continuous. Show that $\{\|A_n\| : n \in \mathbb{N}\}$ is bounded.

**b.** Let $a_j$, $j = 1, \ldots, N$ be a sequence of real numbers with each $a_j > 0$ and $\sum_{j=1}^{N} a_j = A$. Prove that

$$\sum_{j=1}^{N} \frac{1}{a_j} \geq \frac{N^2}{A}.$$

When is equality achieved?

**2** Let $f \in L^1(\mathbb{R})$, and

$$f_\rho(x) = \frac{1}{2\rho} \int_{x-\rho}^{x+\rho} f(t)\, dt, \quad \rho > 0.$$

Show that the function $f_\rho$ is continuous for each $\rho > 0$, and $f_\rho \to f$ in $L^1(\mathbb{R})$ as $\rho \to 0$.

**3** Let $H$ be a separable infinite dimensional Hilbert space, and suppose that $e_1, e_2, \ldots$ is an orthonormal system in $H$. Let $f_1, f_2, \ldots$ be another orthonormal system which is complete, i.e. such that the closure of the span of $\{f_j\}$ is all of $H$.

**a.** Prove that if $\sum_{n=1}^{\infty} ||e_n - f_n||^2 < 1$, then $\{e_n\}$ is also a complete orthonormal system.

**b.** Suppose only that $\sum_{n=1}^{\infty} ||e_n - f_n||^2 < \infty$. Prove that it is still true that $\{e_n\}$ is a complete orthonormal system. (Hint: choose $N$ so that $\sum_{n=N+1}^{\infty} ||e_n - f_n||^2 < 1$; consider the subspace $S$ spanned by the vectors $\tilde{f}_n = f_n - \sum_{k=n+1}^{\infty} \langle f_n, e_n \rangle e_n$, for $n \leq N$.)

**4** Let $c_0$ denote the closed subspace in $\ell^\infty(\mathbb{Z})$ consisting of all bilateral sequences $x = (x_j)$ such that $x_j \to 0$ when $|j| \to \infty$. The sequence of Fourier coefficients $a_n$ of any function $f \in L^1(S^1)$ lies in $c_0$. Denoting this map by $\mathcal{F}$, prove that the image of $\mathcal{F}$ is not all of $c_0$.

**5** Let $\mu$ be a non-negative Borel measure on $\mathbb{R}^n$ such that $\mu(A) < \infty$ for each bounded Borel subset $A \subset \mathbb{R}^n$.

**a.** Setting $B_\rho(x) = \{y : |y - x| < \rho\}$, prove that for each $\rho > 0$, $\mu(B_\rho(x))$ is a lower semi-continuous function on $\mathbb{R}^n$. (A real-valued function $\theta$ on $\mathbb{R}^n$ is lower semi-continuous if for all $x \in \mathbb{R}^n$, $\theta(x) \leq \liminf_{y \to x} \theta(y)$.)

**b.** Prove that the upper $k$-density $\Theta^{*k}(x) \equiv \limsup_{\rho \searrow 0} \rho^{-k} \mu(B_\rho(x))$ is a Borel measurable function on $\mathbb{R}^n$ for each $k > 0$.

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**Spring 2007, part II**

Do all the problems. Write your solution for each problem in a separate blue book.

**1** Let $f : [0,1] \to \mathbb{R}$ be any function of bounded variation. Prove that

$$\int_0^1 |f'(x)|\,dx \le T(f),$$

where $T(f)$ is the total variation of $f$, i.e. the supremum over all partitions

$$0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1$$

of $\sum_{j=1}^N |f(x_j) - f(x_{j-1})|$.

**2** Let $\phi_a = a^{-1} 1_{[-a/2,a/2]}$. Assume that $a_n > 0$, $n = 1, 2, \ldots$, $\sum a_n < \infty$, and let

$$\Phi_n = \phi_{a_1} * \phi_{a_2} * \cdots * \phi_{a_n}.$$

Prove that for each $k = 0, 1, 2 \ldots$, $\Phi_n$ is in $\mathcal{C}^k(\mathbb{R})$ for $n$ sufficiently large, and there is a function $\Phi \in \mathcal{C}^\infty(\mathbb{R})$ which is supported in an interval of length $\sum a_n$, and satisfies $\int \Phi(x)\,dx = 1$, such that for each $k$, $\Phi_n$ converges to $\Phi$ in $\mathcal{C}^k$ (on compact intervals) as $n \to \infty$.

**3** Let $\mathcal{F}$ denote the Fourier transform initially defined as a map from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$.

**a.** Prove that $\mathcal{F}$ extends to a bounded map $L^{4/3}(\mathbb{R}) \longrightarrow L^4(\mathbb{R})$.

**b.** Prove, on the other hand, that $\mathcal{F}$ does not extend to a bounded map $L^4(\mathbb{R}) \longrightarrow L^{4/3}(\mathbb{R})$.

**4** Let $X$ be a Banach space such that $X^*$ is separable. Prove that $X$ is separable.

**5** If $f \in L^1(S^1)$, then we write

$$S_N(f)(x) = \sum_{n=-N}^{N} a_n e^{inx}$$

and

$$\sigma_N(f)(x) = \frac{1}{N} \sum_{M=0}^{N-1} S_M(f)(x)$$

the $N^{\text{th}}$ partial sum and the $N^{\text{th}}$ Fejer sum of its Fourier series, respectively. Prove that if $f, g$ are two smooth functions with $f \le g$, then $\sigma_N(f)(x) \le \sigma_N(g)(x)$ for all $x$, but that there exist functions $f \le g$ such that $S_N(f)(0) > S_N(g)(0)$.