

## Spring 2007 Qual, Part I: 1,3,4; Part II: 4

## I.1

1. Suppose that  $X$  and  $Y$  are Banach spaces, and  $A_n$ ,  $n = 1, 2, 3, \dots$ , and  $A$  are bounded linear operators from  $X$  to  $Y$ . Suppose also that  $A_n \rightarrow A$  in the weak operator topology, i.e. the weakest topology on bounded linear operators  $\mathcal{L}(X, Y)$  in which the maps

$$E_{\ell, x} : \mathcal{L}(X, Y) \ni A \mapsto \ell(Ax), \quad x \in X, \ell \in Y^*,$$

are continuous. Show that  $\{\|A_n\| : n \in \mathbb{N}\}$  is bounded.

2. Let  $a_j$ ,  $j = 1, \dots, N$  be a sequence of real numbers with each  $a_j > 0$  and  $\sum_{j=1}^N a_j = A$ . Prove that

$$\sum_{j=1}^N \frac{1}{a_j} \geq \frac{N^2}{A}.$$

When is equality achieved?

1. We start by regarding  $A_n x \in Y \subset Y^{**}$ . For each  $x \in X$  and  $\ell \in Y^*$ , we have that the sequence  $|\ell(A_n x)|$  converges and so is bounded. The uniform boundedness principle then tells us that the norms  $\|A_n x\|$  are uniformly bounded (for each  $x \in X$ ). We are in a situation where we may apply the uniform boundedness principle again to conclude that the norms  $\|A_n\|$  are uniformly bounded.
2. This problem follows quickly from the observation that if  $x > 0$  is a positive real number, then

$$x + \frac{1}{x} \geq 2,$$

with equality holding if and only if  $x = 1$ . (This is equivalent to  $(x - 1)^2 \geq 0$ .)

Now we have that

$$\begin{aligned} A \sum_{j=1}^N \frac{1}{a_j} &= \sum_{i=1}^N \sum_{j=1}^N \frac{a_i}{a_j} = \sum_{1 \leq i < j \leq N} \left( \frac{a_i}{a_j} + \frac{a_j}{a_i} \right) + \sum_{i=1}^N \frac{a_i}{a_i} \\ &\geq 2 \binom{N}{2} + N = N^2, \end{aligned}$$

with equality holding if and only if  $\frac{a_i}{a_j} = 1$  for all  $i, j$ , i.e. if all  $a_i$  are equal. ■

**I.3** Let  $H$  be a separable infinite dimensional Hilbert space, and suppose that  $e_1, e_2, \dots$  is an orthonormal system in  $H$ . Let  $f_1, f_2, \dots$  be another orthonormal system which is complete, i.e. such that the closure of the span of  $\{f_j\}$  is all of  $H$ .

1. Prove that if  $\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1$ , then  $\{e_n\}$  is also a complete orthonormal system.
2. Suppose only that  $\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \infty$ . Prove that it is still true that  $\{e_n\}$  is a complete orthonormal system.

1. It is enough to show that if  $\langle v, e_n \rangle = 0$  for all  $n$ , then  $v = 0$ . Suppose that we have such a  $v$ . By the completeness of  $\{f_n\}$ , we have

$$\begin{aligned} \|v\|^2 &= \sum_{n=1}^{\infty} |\langle v, f_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle v, f_n - e_n \rangle|^2 \\ &\leq \|v\|^2 \sum_{n=1}^{\infty} \|e_n - f_n\|^2, \end{aligned}$$

which is a contradiction unless  $v = 0$  because  $\sum \|e_n - f_n\|^2 < 1$ .

2. Start by choosing  $N$  such that  $\sum_{n > N} \|e_n - f_n\|^2 < 1$ . For  $n \leq N$ , let

$$\tilde{f}_n = f_n - \sum_{k=n+1}^{\infty} \langle f_n, e_k \rangle e_k.$$

Suppose that  $\langle \tilde{f}_n, v \rangle = 0$  for all  $n \leq N$  and  $\langle e_n, v \rangle = 0$  for all  $n > N$ . Note that this implies that  $\langle f_n, v \rangle = 0$  for all  $n \leq N$ . (It is easy to see that  $\langle f_N, v \rangle = 0$ , and then one can just proceed by induction.) We then have

$$\begin{aligned} \|v\|^2 &= \sum_{n=N+1}^{\infty} |\langle v, f_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle v, f_n - e_n \rangle|^2 \\ &\leq \|v\|^2 \sum_{n=N+1}^{\infty} \|f_n - e_n\|^2, \end{aligned}$$

which again allows us to conclude that  $v = 0$ .

We then have that  $\{e_n : n > N\}$  is a complete orthonormal basis for  $S^\perp$ , where  $S$  is the span of  $\tilde{f}_n, n \leq N$ . This observation also shows that  $e_n \perp S^\perp$  for  $n \leq N$ , so  $e_n \in S$  for  $n \leq N$ . We know already that  $S$  has dimension less than or equal to  $N$ , and  $e_1, \dots, e_N$  are orthonormal, so that  $e_1, \dots, e_N$  form an orthonormal basis of  $S$ . Finally,  $H = S \oplus S^\perp$ , so the  $e_n$  must form a complete orthonormal basis for  $H$ . ■

**I.4** Let  $c_0$  denote the closed subspace in  $\ell^\infty(\mathbb{Z})$  consisting of all bilateral sequences  $x = (x_j)$  such that  $x_j \rightarrow 0$  when  $|j| \rightarrow \infty$ . The sequence of Fourier coefficients  $a_n$  of any function  $f \in L^1(\mathbb{S}^1)$  lies in  $c_0$ . Denoting this map by  $\mathcal{F}$ , prove that the image of  $\mathcal{F}$  is not all of  $c_0$ .

We know already that  $\mathcal{F}$  is bounded and injective. If it were surjective, then the open mapping theorem would tell us that it has a bounded inverse. In particular, bounded sets in  $c_0$  would correspond to bounded sets in  $L^1$ . We will show that this is not the case.

Consider the functions

$$D_N(x) = \sum_{n=-N}^N e^{inx} = \frac{\sin\left((N + \frac{1}{2})x\right)}{\sin\left(\frac{x}{2}\right)}.$$

The Fourier coefficients of  $D_N$  are

$$\widehat{D}_N(n) = \begin{cases} 1 & |n| \leq N \\ 0 & |n| > N \end{cases},$$

and so their Fourier coefficients form a bounded set in  $c_0$ , as  $\|\widehat{D}_N\|_{c_0} \leq 1$  for all  $N$ .

Note that  $D_N(x)$  has zeros at  $\frac{2k\pi}{2N+1}$  (except at  $k = 0$ ), and that we certainly have

$$\|D_N\|_{L^1} \geq \sum_{k=1}^N \int_{\frac{2(k-1)\pi}{2N+1}}^{\frac{2k\pi}{2N+1}} \frac{|\sin(N + \frac{1}{2})x|}{|\sin x/2|} dx,$$

so we need only estimate each one of these terms. We know that  $\sin x/2$  is increasing on  $[0, \pi]$ , so we can write

$$\begin{aligned} \sum_{k=1}^N \int_{\frac{2(k-1)\pi}{2N+1}}^{\frac{2k\pi}{2N+1}} \frac{|\sin(N + \frac{1}{2})x|}{|\sin x/2|} dx &\geq \sum_{k=1}^N \frac{2N+1}{2\pi k} \int_{\frac{2(k-1)\pi}{2N+1}}^{\frac{2k\pi}{2N+1}} |\sin(N + \frac{1}{2})x| dx \\ &= \frac{1}{\pi} \sum_{k=1}^N \frac{1}{k} \left(1 - \cos \frac{2N\pi}{2N+1}\right) \geq c \log N. \end{aligned}$$

This tells us that  $\|D_N\|_{L^1} \geq c \log N$  as  $N \rightarrow \infty$ , and so these do not form a bounded set in  $L^1$ , meaning that  $\mathcal{F}$  cannot be surjective.

Another way of thinking about this problem is as follows: We already know that  $\mathcal{F}$  is a continuous and injective map  $L^1(\mathbb{S}^1) \rightarrow c_0$ . If it were surjective then the open mapping theorem would give us an isomorphism of the two spaces. In particular, it would give us an isomorphism of the dual spaces. We know, however, that  $(L^1(\mathbb{S}^1))^* = L^\infty(\mathbb{S}^1)$ , which is not separable, while  $c_0^* = \ell_1$ , which is separable, so the two spaces cannot be isomorphic. ■

**II.4** Let  $X$  be a Banach space such that  $X^*$  is separable. Prove that  $X$  is separable.

This is in quite a few texts, including Reed and Simon. This solution is taken from that source with notation changed only slightly.

Suppose that  $\ell_n$  is a countable dense set in  $X^*$ . Let  $x_n \in X$  be such that  $\|x_n\| = 1$  and  $|\ell_n(x_n)| \geq \|\ell_n\|/2$ . Consider the set  $S$  of all finite linear combinations of the  $x_n$  with rational coefficients.  $S$  is then a countable union of countable sets, so  $S$  is countable. We claim that  $S$  is dense in  $X$ .

Suppose that  $S$  is not dense, so there is some  $y \in X, y \notin \bar{S}$ , the closure of  $S$ . Then there must be some linear functional  $\ell \in X^*$  such that  $\ell = 0$  on  $S$  but  $\ell(y) \neq 0$ . Let  $\ell_{n(k)}$  be a subsequence of  $\ell_n$  such that  $\ell_{n(k)} \rightarrow \ell$  in  $X^*$ . We then have, by our choice of  $x_n$ , that

$$\|\ell - \ell_{n(k)}\|_{X^*} \geq |(\ell - \ell_{n(k)})(x_{n(k)})| = |\ell_{n(k)}(x_{n(k)})| \geq \|\ell_{n(k)}\|/2.$$

The left hand side of the above inequality tends to 0, so we must have that  $\|\ell_{n(k)}\| \rightarrow 0$  as  $k \rightarrow \infty$ , and so  $\ell = 0$ . This contradicts that  $\ell(y) \neq 0$ . ■