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Qual Problems, Fall 2005

Fall 2005 Qual, Part I: 1b, 3, 4; Part II: 1, 4, 5

I.1b Suppose that u is a continuous linear functional on $C^\infty(\mathbb{T})$ (i.e. a distribution on the circle), which has the property that $\langle u, \phi \rangle \geq 0$ whenever $\phi \geq 0$, $\phi \in C^\infty(\mathbb{T})$. Show that u is a measure.

We know that the dual of $C(\mathbb{T})$ is the set of finite measures on $\mathbb{T}$, so it suffices to show that u extends to a linear functional on $C(\mathbb{T})$. We know that u is linear, so it suffices to show that u is bounded with respect to the $C^0(\mathbb{T})$ norm, i.e.

$$|\langle u, \phi \rangle| \leq C \sup |\phi|.$$

Let $\phi \in C^\infty(\mathbb{T})$. Let $M = \sup |\phi|$, and let $\psi_1 = \Re(\phi)$, $\psi_2 = \Im(\phi)$ be the real and imaginary parts of ϕ. Observe that $|\psi_i| \leq M$ for $i = 1, 2$, so we have that $M \pm \psi_i \geq 0$ for $i = 1, 2$. The hypothesis on u then gives us that

$$\langle u, M - \psi_i \rangle \geq 0, \ \langle u, M + \psi_i \rangle \geq 0.$$

In other words, we have that

$$\pm \langle u, \psi_i \rangle \leq M \langle u, 1 \rangle.$$

Let $C = \langle u, 1 \rangle$. The above inequality may then be rewritten as

$$|\langle u, \psi_i \rangle| \leq CM$$

because $\langle u, \psi_i \rangle$ is real. We then have that

$$\begin{aligned} |\langle u, \phi \rangle| &\leq |\langle u, \psi_1 \rangle| + |\langle u, \psi_2 \rangle| \\ &\leq CM + CM \leq 2C \|\phi\|_{C^0(\mathbb{T})}. \end{aligned}$$

This tells us that $u \in \left(C^0(\mathbb{T})\right)^*$ and so is a measure. ∎

I.3 Show that if $g \in L^1(\mathbb{T})$, $\mu \in M(\mathbb{T})$ (a finite measure on $\mathbb{T}$), and $\mu(x + \alpha\pi) - \mu(x) = g dt$, for some irrational α, then μ is absolutely continuous.

We begin by decomposing $\mu = \mu_{ac} + \mu_s$ into its absolutely continuous and singular parts. We wish to show that $\mu_s = 0$.

By adding $\mu(x)$ to both sides, we see that

$$\mu_{ac}(t + \alpha\pi) + \mu_s(t + \alpha\pi) = (\mu_{ac}(t) + g(t)dt) + \mu_s(t).$$

The uniqueness of this decomposition means that the singular parts of the two sides must be equal, i.e.

$$\mu_s(t + \alpha\pi) = \mu_s(t).$$

Measures on $\mathbb{T}$ are certainly distributions on $\mathbb{T}$, so we may consider the Fourier coefficients of μ_s. Recall that if $u_\tau = u(t - \tau)$, then

$$\hat{u_\tau}(n) = e^{in\tau}\hat{u}(n).$$

We must thus have that

$$e^{-in\alpha\pi}\hat{\mu_s}(n) = \hat{\mu_s}(n)$$

for all n. α is irrational, so $e^{-in\alpha\pi} \neq 1$ for all n, and so we must have that $\hat{\mu_s}(n) = 0$ for all n, i.e. $\mu_s = 0$. Thus μ is absolutely continuous. ∎

I.4 Let $f \in C^\infty(\mathbb{R})$ (the space of infinitely differentiable functions on the line). Assume that for every $x \in \mathbb{R}$, $f^{(n)}(x) = 0$ for at least one $n \geq 0$. Prove that f is a polynomial.

The outline of our solution is as follows: First we will use the Baire category theorem to show the existence of a dense open set U such that the restriction of f to any component of this set agrees with a polynomial. We will then show that the complement F of this set has no isolated points. We will use Baire again to show that there is some open interval J such that $J \cap F \subset U$, which will give us a contradiction.

Start by considering the sets

$$F_n\{x \in \mathbb{R} : f^{(n)}(x) = 0\}.$$

F_n is closed because f is smooth, and $\mathbb{R} = \cup_n F_n$ by hypothesis. Let $U = \cup_n F_n^\circ$, i.e. U is the union of all the neighborhoods in $\mathbb{R}$ on which f agrees with a polynomial. Differentiation is a local property, so $F_n^\circ \subset F_{n+1}^\circ$. U is clearly open. For any interval $[a, b]$, we may write

$$[a, b] = \cup F_n \cap [a, b].$$

The Baire category theorem tells us that there is some n such that $F_n \cap [a, b]$ has nonempty interior, i.e. $U \cap [a, b]$ is nonempty. Thus U is dense.

Suppose that $J \subseteq U$ is a component interval of U (we know already that the connected components of an open set in $\mathbb{R}$ are intervals). Let K_i be an exhaustion of J by compact intervals, i.e. K_i is a compact interval, $K_i \subset K_{i+1}$ and $J = \cup K_i$. (For example, if $J = (a, b)$, then let $K_i = [a + \frac{1}{i}, b - \frac{1}{i}]$.) K_i is compact and the F_n° provide an open cover of K_i, so there is some finite subcover. The F_n° are nested, so there is some n_i such that

$$K_i \subset F_{n_i}^\circ,$$

i.e. f is a polynomial of degree at most n_i on K_i. The polynomial $f|_{K_i}$ is determined by at most $n_i + 1$ points, which we may take to be in K_1, so f is a polynomial of degree less than n_1 on all of K_i, and so on J. In particular, if $U = \mathbb{R}$, then f is a polynomial on I and we are done.

Our goal then is to show that the nowhere dense closed set $F = \mathbb{R} \backslash U$ is empty. We'll start by showing it has no isolated points. Indeed, if $x \in F$ is an isolated point, then there must be intervals (a, x) and (x, b) in U. f is a polynomial when restricted to those intervals, and so if we take n high enough, $f^{(n)}$ vanishes on (a, x) and (x, b). Continuity then implies that $f^{(n)}(x) = 0$ as well, so in fact $(a, b) \subset U$.

F is thus a complete metric space (because it is closed) with no isolated points. We apply Baire again to the decomposition $F = \bigcup_n F_n \cap F$. This tells us that there is an open interval J such that

$$\emptyset \neq J \cap F \subseteq F_n.$$

Moreover, by using a sequence of points $x_k \in F$ tending to $x \in J \cap F$ (because F has no isolated points), we may see that $J \cap F \subseteq F_{n+k}$ for all $k \geq 0$.

We finally claim that $J \subseteq F_n$, which would imply that $J \subset U$, contradicting that $J \cap F \neq \emptyset$. It is enough to show that each component L of $J \cap U$ lies in F_n°. L is a component of $J \cap U$, so L is an interval contained in U, meaning that $L \subset F_m^\circ$ for some minimal m. Suppose that $m > n$. Then by integrating, we can see that $f^{(m-1)}$ is constant on L. Note that L is a component of $J \cap U$, so $f^{(n)}$ vanishes on the endpoints of L. In particular, if $m > n$, $f^{(m-1)}$ must vanish on L by continuity, contradicting the minimality of m. Thus $m \leq n$, and so $L \subset F_m^\circ \subset F_n^\circ$.

We may thus conclude that F is empty, and so f is a polynomial. ■

II.1

1. Describe a norm $\|\cdot\|_0$ on $\mathbb{R}^3$ such that the unit vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ have norm 1 while $\|(1,1,1)\|_0 < \frac{1}{100}$.
2. Let $f_n(t) = \sum_{j \in \mathbb{Z}} \hat{f}_n(j) e^{ijt}$, where $|\hat{f}_n(j)| \leq |j|^{-\log|j|}$ for $|j| > 75$, uniformly in n. Assume that for all j, $\lim_n \hat{f}_n(j)$ exists, and denote it c_j. Prove that $g = \sum_j c_j e^{ijt} \in C^\infty(\mathbb{T})$ and that f_n converges to g in the topology of C^k for every $k > 0$.

1. The idea here is to note that the unit ball must be convex (because we are looking for a norm) and must contain $(100, 100, 100)$, while the standard basis vectors must have norm 1.

   We consider an ℓ^∞-type norm. Indeed, consider the vectors $v_1 = (1,0,0)$, $v_2 = (0,1,0)$, and $v_3 = (200, 200, 200)$. This is a basis for $\mathbb{R}^3$, so any vector in $\mathbb{R}^3$ may be expressed uniquely as a linear combination of these three. We may thus specify a norm by letting

   $$\|a_1 v_1 + a_2 v_2 + a_3 v_3\|_0 = \sup_{i=1,2,3} |a_i|.$$

   We then note that

   $$(0,0,1) = \frac{1}{200} v_3 - v_1 - v_2,$$

   so that $\|(0,0,1)\|_0 = 1$, while $\|(1,1,1)\|_0 = \frac{1}{200}$.

2. Recall that to see that $g \in C^\infty$, it is enough to show that

$$\sum_{j \in \mathbb{Z}} |j|^k |c_j| < \infty$$

for all k. Our hypothesis is that

$$|\hat{f}_n(j)| \leq |j|^{-\log|j|}$$

for all $|j| > 75$, uniformly in n, so we may pass to the limit and observe that c_j satisfies the same bounds. $\log|j|$ is unbounded, so it is easy to see that for some C_k depending on k,

$$\sum |j|^k |c_j| \leq C_k + \sum_{\log|j| > k+2} |j|^{-\log|j| + k} < \infty,$$

and so $g \in C^\infty(\mathbb{T})$.

To see that $f_n \to g$ uniformly, it is enough to obtain similar bounds on $|c_j - \hat{f}_n(j)|$, i.e. that

$$\sum_{j \in \mathbb{Z}} |j|^k |c_j - \hat{f}_n(j)| \to 0$$

as $n \to \infty$. Fix $\epsilon > 0$. By taking N large enough, we may assume that

$$\sum_{|j| > N} |j|^k |c_j - \hat{f}_n(j)| < \frac{\epsilon}{2}.$$

By taking n large enough, we may also guarantee that

$$\sum_{|j| \leq N} |j|^k |c_j - \hat{f}_n(j)| < \frac{\epsilon}{2},$$

proving the claim.

■

II.4 Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the circle group. Let $k \in L^1(\mathbb{T})$ and let K be the integral operator on $L^2(\mathbb{T})$ defined by

$$K : f \mapsto \frac{1}{2\pi} \int k(x-t) f(t) dt.$$

1. Prove that K is compact and normal (i.e. commutes with its adjoint). When is it actually self-adjoint?

2. What is the spectrum of K and what are the corresponding eigenfunctions and eigenvalues?

3. If we replace $\mathbb{T}$ by $\mathbb{R}$, then is the analogous operator on $L^2(\mathbb{R})$ (with $k \in L^1(\mathbb{R})$) compact?

1. To see that K is compact, we use Fourier series. Observe that for $f \in L^2$, we have that

$$\begin{aligned}\widehat{Kf}(n) &= \frac{1}{(2\pi)^2}\int\int e^{-inx}k(x-t)f(t)dtdx \\ &= \frac{1}{(2\pi)^2}\int\int e^{-in(x-t)}e^{-int}k(x-t)f(t)dtdx \\ &= \frac{1}{(2\pi)^2}\left(\int e^{-iny}k(y)dy\right)\left(\int e^{-int}f(t)dt\right) \\ &= \hat{k}(n)\hat{f}(n).\end{aligned}$$

Thus we may consider K as acting on $\ell^2(\mathbb{Z})$ (which is isometrically isomorphic to $L^2(\mathbb{T})$) via multiplication by $\hat{k}(n)$.

$k \in L^1(\mathbb{T})$, so the Riemann-Lebesgue lemma tells us that $\hat{k}(n) \to 0$ as $|n| \to \infty$. Consider the operators K_N, which act on L^2 by

$$\widehat{K_N f}(n) = \begin{cases} \hat{k}(n)\hat{f}(n) & |n| \le N \\ 0 & |n| > N \end{cases}.$$

If N is large, then $|\hat{k}(n)| < \epsilon$ for $|n| > N$. In this case, we have

$$\begin{aligned}\|(K-K_N)f\|_{L^2}^2 &= \|(\widehat{K-K_N})f\|_{\ell^2}^2 \\ &= \sum_{|n|>N}\left|\hat{k}(n)\right|^2\left|\hat{f}(n)\right|^2 \\ &\le \epsilon\sum\left|\hat{f}(n)\right|^2 \le \epsilon\|f\|_{L^2}^2,\end{aligned}$$

so that $\|K - K_N\| < \epsilon$. Thus K may be approximated in norm by finite rank operators and so is compact.

We may also see that K is normal by using Fourier series. Indeed, observe that for $f, g \in L^2(\mathbb{T})$, we have

$$\begin{aligned}\langle Kf, g\rangle_{L^2} &= \left\langle \widehat{Kf}, g\right\rangle_{\ell^2} \\ &= \sum \hat{k}(n)\hat{f}(n)\overline{\hat{g}(n)},\end{aligned}$$

so that

$$\widehat{K^*g}(n) = \overline{\hat{k}(n)}\hat{g}(n).$$

To see that K is normal, we simply observe that

$$\widehat{KK^*f}(n) = \hat{k}(n)\overline{\hat{k}(n)}\hat{f}(n) = \overline{\hat{k}(n)}\hat{k}(n)\hat{f}(n) = \widehat{K^*Kf}(n),$$

and so $KK^*f = K^*Kf$ for all $f \in L^2$. Finally, note that K will then be self-adjoint whenever $\hat{k}(n) = \overline{\hat{k}(n)}$, i.e. whenever $\hat{k}(n)$ is real. Note that this requirement can be rephrased as

$$\int e^{-inx}k(x)dx = \overline{\int e^{-inx}k(x)dx} = \int e^{inx}\overline{k}(x)dx = \int e^{-inx}\overline{k(-x)}dx.$$

The density of trigonometric polynomials in L^1 mean that we must have that $k(x) = \overline{k(-x)}$ in L^1.

2. Let $\lambda_n = \hat{k}(n)$. Note that we must have $\lambda_n \in \sigma(K)$, as

$$Ke^{inx} = \hat{k}(n)e^{inx} = \lambda_n e^{inx}.$$

Moreover, because $\hat{k}(n) \to 0$, $0 \in \sigma(K)$ because it is closed. We claim that this is the entire spectrum. Note that this also implies that 0 is the only limit point of $\{\hat{k}(n)\}$. Suppose now that

$$\lambda \notin \{0, \lambda_n\}.$$

This set is closed (it contains all of its limit points), so there is some $\epsilon > 0$ such that $|\lambda - \lambda_n| \geq \epsilon$ for all n. Consider the map on ℓ^2 defined by

$$a_n \mapsto \frac{1}{\lambda - \lambda_n} a_n.$$

This is bounded because

$$\left| \frac{1}{\lambda - \lambda_n} \right| \leq \frac{1}{\epsilon}.$$

Using the isometry between L^2 and ℓ^2, we define the bounded operator A_λ by

$$\widehat{Af}(n) = \frac{1}{\lambda - \lambda_n}\hat{f}(n).$$

Note that we then have

$$A\widehat{(\lambda - K)}f(n) = \hat{f}(n), \; \widehat{(\lambda - K)}Af(n) = \hat{f}(n),$$

and so $A = (\lambda - K)^{-1}$, i.e. $\lambda \notin \sigma(K)$.

3. In terms of intuition, it is important to think about what we used to prove K was compact. We relied very heavily on our representation of K as a multiplier on $\ell^2(\mathbb{Z})$ and that such operators are compact if and only if they tend to 0. The analogous fact is not true on $\mathbb{R}$. While it is true that we can represent the analogous operator as a multiplier on $L^2(\mathbb{R})$ using the Fourier transform, we only know that it is multiplication by a continuous function tending to 0 at ∞.

To be (somewhat) more rigorous, we fix a smooth compactly supported function that is identically 1 on $[-1, 1]$. Let k be the inverse Fourier transform of this function.

We know that such a function is Schwartz and so is certainly L^1. The analogous operator then is

$$K : f \mapsto \frac{1}{2\pi} \int_{\mathbb{R}} k(x-t) f(t) dt.$$

The Fourier transform gives us an isomorphism of $L^2(\mathbb{R})$, and we may write

$$\widehat{Kf}(\xi) = \hat{k}(\xi)\hat{f}(\xi).$$

Note that K is compact on $L^2(\mathbb{R})$ if and only if this multiplier (call it A) is compact on $L^2(\mathbb{R})$. We claim that it is not. Indeed, consider the characteristic functions

$$f_n(\xi) = \sqrt{n}\chi_{[-1/2n, 1/2n]}(\xi) \in L^2(\mathbb{R}).$$

Note that $\|f_n\|_{L^2} = 1$, so this is a bounded sequence and that $Af_n = f_n$ (because $\hat{k}$ is constant on $[-1,1]$). Moreover, this sequence can have no bounded subsequence because $f_n \to 0$ a.e., so that any convergent subsequence must converge to 0, which is impossible because $\|f_n\| = 1$. Thus A is not compact and so K is not compact.

∎

II.5 Suppose that for some p, $1 < p < \infty$, $f_n \in L^p([0,1])$ and $\|f_n\|_p \leq 1$, uniformly in n. Assuming that $f_n(x) \to 0$ a.e., prove that $f_n \to 0$ weakly in L^p.

To show that $f_n \to 0$ weakly in L^p, it is enough to show that $\int_0^1 f_n g \to 0$ for each $g \in L^q([0,1])$, as L^q is the dual of L^p.

Let $g \in L^q$ and fix $\epsilon > 0$. Recall from measure theory that there is some $\delta > 0$ such that if $A \subset [0,1]$ has $m(A) < \delta$, then

$$\left(\int_A |g|^q \right)^{1/q} < \frac{\epsilon}{2}.$$

$f_n \to 0$ almost everywhere, so by Lusin's theorem there is a set $A \subset [0,1]$ with $m(A) < \delta$ such that $f_n \to 0$ uniformly on $[0,1] \backslash A$. In other words, there is some N such that for all $n \geq N$ we have

$$|f_n(x)| \leq \frac{\epsilon}{2\|g\|_q} \text{ on } [0,1] \backslash A.$$

We then have that

$$\begin{aligned}
|\langle f_n, g \rangle| &\leq \int_A |f_n g| + \int_{[0,1] \backslash A} |f_n g| \\
&\leq \|f_n\|_p \left(\int_A |g|^q \right)^{1/q} + \left(\int_{[0,1] \backslash A} |f_n|^p \right)^{1/p} \|g\|_q \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2\|g\|_q} \cdot \|g\|_q = \epsilon.
\end{aligned}$$

∎