Ph.D. Qualifying Exam, Real Analysis

Fall 2007, part I

Do all the problems. Write your solution for each problem in a separate blue book.

1 Two short problems:

a. Suppose that $(X, \mathcal{A}, \mu)$ is a measure space, and $X = \cup_{j=1}^{\infty} X_j$ with $X_j \subset X_{j+1}$ for all j. Let χ_j be the indicator (characteristic) function of X_j, and let M_j denote the multiplication operator by χ_j acting on $Z = L^p(X, d\mu)$, $1 \leq p < \infty$. Show that $M_j \to I$, the identity operator, in the strong operator topology, i.e. the weakest topology on bounded linear operators $\mathcal{L}(Z)$ in which the maps

$$E_z : \mathcal{L}(Z) \ni A \mapsto Az \in Z, \ z \in Z,$$

are continuous, but not necessarily in the norm topology.

b. For $f \in L^1(\mathbb{R})$ let $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx$ denote the Fourier transform of f.

Show that if f has compact support (i.e. has a representative vanishing outside a compact set) then $\hat{f}$ extends to an analytic function on all of $\mathbb{C}$.

2 If μ is a σ-finite measure on a measurable space $(X, \mathcal{A})$ (i.e. on $\mathcal{A}$, where $\mathcal{A}$ is a σ-algebra of subsets of X), then there is a *finite* measure ν on $(X, \mathcal{A})$ with $\nu << \mu$ and $\mu << \nu$.

3 Suppose that H_1 and H_2 are separable Hilbert spaces and $A : H_1 \to H_2$ is a bounded linear operator. Suppose that there exist $B \in \mathcal{L}(H_2, H_1)$ and compact operators E_j on H_j, $j = 1, 2$, such that $BA = I_1 - E_1$, $AB = I_2 - E_2$, where I_j is the identity operator on H_j. Show that the nullspace of A is finite dimensional, the range of A is closed in H_2, and its orthocomplement is finite dimensional.

4 Suppose that $\{a_n : \ n = 0, 1, 2, \ldots\}$ is *any* sequence of real numbers. Show that there exists a real valued function $f \in C^\infty(\mathbb{R})$ such that $f^{(n)}(0) = a_n$. (Hint: let $\chi \in C_c^\infty(\mathbb{R})$ identically 1 near 0, and choose $\epsilon_n > 0$ appropriately so that $\lim_{n\to\infty} \epsilon_n = 0$ and

$$\sum_{n=0}^{\infty} \frac{a_n x^n}{n!}\, \chi(x/\epsilon_n)$$

converges in C^∞.)

5 Let $L^2([0, 1])$ denote the Hilbert space of complex valued square integrable functions on $[0, 1]$ with the usual inner product $\langle f, g\rangle = \int_0^1 f(x)\, \overline{g(x)}\, dx$. Define $T : L^2([0, 1]) \to L^2([0, 1])$ by $Tf(x) = \int_0^x f(t)\, dt$ for $x \in [0, 1]$.

a. Show that T is bounded and compact.

b. Show that T has no eigenvalues, i.e. $Tf = \lambda f$, $\lambda \in \mathbb{C}$, $f \in L^2([0, 1])$, implies $f = 0$.

c. Find $\lim_{n\to\infty} \|T^n\|$, and using this or otherwise prove that the spectrum of T is $\{0\}$, i.e. $T - \lambda I$ is an isomorphism of $L^2([0, 1])$ onto itself if and only if $\lambda \neq 0$.

Ph.D. Qualifying Exam, Real Analysis

Fall 2007, part II

Do all the problems. Write your solution for each problem in a separate blue book.

1 Let $C([0,1])$ denote the Banach space of real valued continuous functions on $[0,1]$ with the $\sup$ norm, and suppose that $X \subset C([0,1])$ is a dense linear subspace. Suppose $\ell : X \to \mathbb{R}$ is a linear map (*not* assumed to be continuous in any sense) such that $\ell(f) \geq 0$ if $f \in X$ and $f \geq 0$. Show that there is a unique Borel measure μ on $[0,1]$ such that $\ell(f) = \int f\,d\mu$ for all $f \in X$.

2 Prove that in the Banach space $C([0,1])$, the C^1 functions form a set of the first category.

3 Find a Lebesgue measurable function $f : [0,1] \to [0,1]$ that is not the a.e. limit of any monotone sequence of continuous functions, i.e. f is such that there are no monotone sequences $f_k : [0,1] \to [0,1]$ of continuous functions such that $\lim_{k\to\infty} f_k = f$ a.e.

(Hint: Show that if $E \subset [0,1]$ is measurable and both E and its complement have positive measure in every interval in $[0,1]$, then the indicator (characteristic) function of E has the property. Then construct such a set E.)

4 Let H be a separable infinite dimensional Hilbert space, and suppose that $e_1, e_2, \ldots$, resp. $f_1, f_2, \ldots$, are orthonormal systems in H. Assume that $\{\lambda_n : n \in \mathbb{N}\}$ is a bounded sequence of complex numbers, and let

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle f_n.$$

Show that

a. T is a bounded linear operator on H, and $\|T\| = \sup_{n\in\mathbb{N}} |\lambda_n|$.

b. T is compact if and only if $\lambda_n \to 0$ as $n \to \infty$.

c. if K is a compact operator on H then there exist orthonormal systems $\{e_n : n \in \mathbb{N}\}$ and $\{f_n : n \in \mathbb{N}\}$ and a sequence $\{\lambda_n : n \in \mathbb{N}\}$ of complex numbers converging to 0 such that $Kx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle f_n$ for all $x \in H$. (Hint: consider first the self-adjoint operator K^*K.)

5 For $s \geq 0$, let $H^s(\mathbb{T})$ be the space of L^2 functions f on the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ whose Fourier coefficients $\hat{f}_n = \int e^{-inx} f(x)\,dx$ satisfy $\sum (1+n^2)^s |\hat{f}_n|^2 < \infty$, with norm $\|f\|_s^2 = (2\pi)^{-1} \sum (1+n^2)^s |\hat{f}_n|^2$.

a. Show that for $r > s \geq 0$, the inclusion map $\iota : H^r(\mathbb{T}) \to H^s(\mathbb{T})$ is compact.

b. Show that if $s > 1/2$, then $H^s(\mathbb{T})$ includes continuously into $C(\mathbb{T})$, and indeed the inclusion map is compact.