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Qual Problems, Fall 2007

Fall 2007 Qual, Part I: 1,3,5; Part II:1,2,4,5

I.1 Two short problems:

1. Suppose that $(X, \mathcal{A}, \mu)$ is a measure space, and $X = \cup_{j=1}^{\infty} X_j$ with $X_j \subset X_{j+1}$ for all j. Let χ_j be the indicator (characteristic) function of X_j, and let M_j denote the multiplication operator by χ_j acting on $Z = L^p(X, d\mu)$, $1 \leq p < \infty$. Show that $M_j \to I$, the identity operator, in the strong operator topology, i.e. the weakest topology on bounded linear operators $\mathcal{L}(Z)$ in which the maps

$$E_z : \mathcal{L}(Z) \ni A \mapsto Az \in Z, z \in Z,$$

are continuous, but not necessarily in the norm topology.

2. For $f \in L^1(\mathbb{R})$, let $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx$ denote the Fourier transform of f. Show that if f has compact support (i.e. has a representative vanishing outside a compact set), then $\hat{f}$ extends to an analytic function on all of $\mathbb{C}$.

1. To show that $M_j \to I$ in the strong operator topology, it is enough to show that $M_j f \to f$ for all $f \in Z$, i.e. that $\|M_j f - f\|_p \to 0$ for each f.

   Suppose first that $\mu(X) < \infty$. Let $\epsilon > 0$. Recall from measure theory that for $f \in L^p(X, d\mu)$, there is a $\delta > 0$ such that if $\mu(A) < \delta$, then

$$\left(\int_A |f|^p \right)^{1/p} < \epsilon.$$

   (Note that δ depends on f.) We know that $X = \cup X_j$, so $\mu(X) = \lim \mu(X_j)$. $\mu(X) < \infty$, so we know that $\mu(X \backslash X_j) \to 0$. In particular, for large enough j, $\mu(X \backslash X_j) < \delta$, and so

$$\|\chi_j f - f\|_p = \left(\int_{X \backslash X_j} |f|^p \right)^{1/p} < \epsilon.$$

   Now suppose that $\mu(X) = \infty$. Given $f \in L^p(X)$ and $\epsilon > 0$, we know from measure theory that there is some measurable set A of finite measure such that

$$\left(\int_{X \backslash A} |f|^p \right)^{1/p} < \epsilon/2.$$

   The argument from above works on the measure space (A, μ), and so we may conclude that for large j,

$$\|\chi_j f - f\|_p \leq \left(\int_{A \backslash (X_j \cap A)} |f|^p \right)^{1/p} + \left(\int_{(X \backslash A) \backslash X_j} |f|^p \right)^{1/p} \leq \epsilon/2 + \left(\int_{X \backslash A} |f|^p \right)^{1/p} \leq \epsilon.$$

We may thus conclude that $M_j \to I$ in the strong operator topology.

To see that this convergence is not necessarily in norm, we just consider the example of $X = \mathbb{R}$ with Lebesgue measure μ, and $X_j = [-j, j]$. It suffices to find, for each j, $f_j \in L^p(\mathbb{R})$ such that $\|\chi_j f_j - f_j\|_p = \|f\|_p$. This will show that $\|M_j - I\| \geq 1$ for each j. This is easy enough – let $f_j(x) = \chi_{(j,j+1)}$, so that $\|f_j\|_p = 1$ and $\chi_j f = 0$. Thus $\|\chi_j f - f\|_p = \|f\|_p$.

2. Suppose f has compact support. Define, for $z \in \mathbb{C}$,

$$F(z) = \int_{\mathbb{R}} e^{-ixz} f(x) dx.$$

We first claim that this defines a continuous function. Suppose that $\operatorname{supp} f \subset [-R, R]$. Then for $x \in \operatorname{supp} f$,

$$|e^{-ixz}| \leq e^{R|\Im z|},$$

so that if $h \to 0$,

$$|F(z+h) - F(z)| = \left| \int_{\mathbb{R}} e^{-ixz}(e^{-ixh} - 1) f(x) dx \right|.$$

Note that, for small $|h|$,

$$\left| e^{-ixz}(e^{-ixh} - 1) f(x) \right| \leq (R|h|) e^{R(|\Im z| + 1)} |f(x)| \to 0,$$

so

$$\left| e^{-ixz}(e^{-ixh} - 1) f(x) \right| \leq R e^{R(|\Im z| + 1)} |f(x)|,$$

which is integrable and so the dominated convergence theorem tells us that F is continuous.

We now claim that F is analytic. Let

$$g(z) = \int_{\mathbb{R}} (-ix) e^{-ixz} f(x) dx.$$

Note that g makes sense because f has compact support. To show that F is analytic, we only need to show that

$$\lim_{|h| \to 0} \frac{|F(z+h) - F(z) - g(z)|}{|h|} = 0$$

for all $z \in \mathbb{C}$. Let $z \in \mathbb{C}$, and consider

$$\frac{|F(z+h) - F(z) - hg(z)|}{|h|} = \frac{1}{|h|} \left| \int_{\mathbb{R}} (e^{-ixh} - 1 + ixh) e^{-ixz} f(x) dx \right|.$$

Note that for $|x| < R$, we have

$$\frac{|e^{-ixh} - 1 + ixh|}{|h|} \to 0$$

as $h \to 0$ because ix is the derivative of e^{-ixz} at $z = 0$. Moreover,

$$\frac{1}{|h|}|(e^{-ixh} - 1 + ixh)e^{-ixz}f(x)| \le Ce^{R(|\Im z|+1)}|f(x)|,$$

which is integrable, and so the dominated convergence theorem tells us that, in fact,

$$\frac{|F(z+h) - F(z) - g(z)|}{|h|} \to 0$$

as $h \to 0$.

■

I.3 Suppose that H_1 and H_2 are separable Hilbert spaces, and $A : H_1 \to H_2$ is a bounded linear operator. Suppose that there exist $B \in \mathcal{L}(H_2, H_1)$ and compact operators E_j on H_j, $j = 1, 2$ such that $BA = I_1 - E_1, AB = I_2 - E_2$, where I_j is the identity operator on H_j. Show that the nullspace of A is finite dimensional, the range of A is closed in H_2, and its orthocomplement is finite dimensional.

First we observe the useful fact that if the unit ball of a Hilbert space X is compact, then X is finite dimensional. Indeed, if not, then we could choose an infinite orthonormal set e_n, contradicting the compactness of the unit ball.

Consider the nullspace $N(A)$ of A. For all $x \in N(A)$, $0 = BAx = I_1x - E_1x$, i.e. $E_1|_{N(A)} = I_1|_{N(A)}$, and so the identity operator is compact on $N(A)$. In particular, the unit ball in $N(A)$ is compact, and so $N(A)$ must be finite dimensional. ($N(A)$ is a Hilbert space because it is closed.)

To see that the range of A is closed, we suppose that $x_n \in H_1$ and $y_n = Ax_n \to y \in H_2$. Suppose first that $\|x_n\|$ is bounded, and then E_1x_n has a convergent subsequence by compactness. Passing to a convergent subsequence $x_{n(j)}$ such that $E_1x_{n(j)} \to z \in H_1$, we have that

$$x_{n(j)} = I_1x_{n(j)} = BAx_{n(j)} + E_1x_{n(j)} \to By + z,$$

so then, by continuity,

$$y = \lim_{j\to\infty} Ax_{n(j)} = A(\lim_{j\to\infty} x_{n(j)}) = A(By + z),$$

so that y is in the range of A.

Now suppose that $\|x_n\|$ is unbounded. We may assume that $x_n \perp \ker A$. Consider $z_n = \frac{x_n}{\|x_n\|}$. (We may clearly also assume that all $\|x_n\| \to \infty$.) Note that because $\|x_n\| \to \infty$, $Az_n = \frac{1}{\|x_n\|}Ax_n \to 0$. We may then repeat the above argument to find a convergent subsequence $z_{n(j)} \to z$. $\|\cdot\|$ is continuous, so $\|z\| = 1$, while $(\ker A)^\perp$ is closed, so $z \perp \ker A$. Moreover, the continuity of A implies $Az = 0$, i.e. $z \in \ker A$, a contradiction because $\|z\| = 1$. Thus A has closed range.

Finally, consider the orthocomplement of the range of A, $(\operatorname{ran} A)^\perp$. By taking the adjoint of the equation $AB = I_2 - E_2$, we have $B^*A^* = I_2 - E_2^*$. There are two things to observe here: that $(\operatorname{ran} A)^{\perp} = \ker A^*$ and that the compactness of E_2 implies the compactness of E_2^*. We may thus apply the same argument as above to A^* to see that $N(A^*) = (\operatorname{ran} A)^{\perp}$ is finite dimensional.

Alternatively, we can approximate E_1 by a finite rank operator F so that $\|E_1 - F\| < 1/2$, and so $I - (E_1 - F)$ is invertible. We can then observe that

$$(I - (E_1 - F))^{-1} BA = (I - (E_1 - F))^{-1}(I - E_1) = I - F.$$

Then if $u \in N(A)$, we have that $0 = u - Fu$, and so u is in the range of F, a finite dimensional subspace. Thus $N(A)$ is finite dimensional. A similar statement about A^* gives that $(\operatorname{ran} A)^{\perp}$ is finite dimensional.

Replacing B by $(I - (E_1 - F))^{-1}B$ as above, we may assume that E_1 is finite rank. Then if we have that $y_n = Ax_n \to y \in H_2$, we must have

$$x_n - Fx_n = By_n \to By.$$

A similar argument to above would then give that y must be in the range of A. ■

I.5 Let $L^2([0,1])$ denote the Hilbert space of complex valued square integrable functions on $[0,1]$ with the usual inner product $(f,g) = \int_0^1 f(x)\overline{g(x)}dx$. Define $T : L^2([0,1]) \to L^2([0,1])$ by $Tf(x) = \int_0^x f(t)dt$ for $x \in [0,1]$.

1. Show that T is bounded and compact.
2. Show that T has no eigenvalues, i.e. $Tf = \lambda f$, $\lambda \in \mathbb{C}$, $f \in L^2([0,1])$, implies that $f = 0$.
3. Find $\lim_{n\to\infty} \|T^n\|$, and using this or otherwise prove that the spectrum of T is $\{0\}$, i.e. $T - \lambda I$ is an isomorphism of $L^2([0,1])$ onto itself if and only if $\lambda \neq 0$.

1. We first show that T is bounded. This is essentially just Cauchy-Schwarz:

$$\begin{aligned}\|Tf\|_2^2 = \int_0^1 \left| \int_0^x f(t)dt \right|^2 dx &\leq \int_0^1 \left(\left(\int_0^x |f(t)|^2 dt \right)^{1/2} (\int_0^x dt)^{1/2} \right)^2 dx \\ &\leq \int_0^1 \int_0^1 |f(t)|^2 dt dx = \|f\|_2^2.\end{aligned}$$

   We also claim that T is a compact operator. Indeed, suppose that $\{f_n\}$ is a bounded sequence in L^2, i.e. the norms $\|f_n\|_2$ are bounded. Our goal is to use Arzela-Ascoli to show that $\{Tf_n\}$ has a uniformly convergent subsequence and hence an L^2-convergent subsequence. This will show that T is a compact operator.

   We need to show $\{Tf\}$ is a uniformly bounded sequence of continuous functions and that it is an equicontinuous family. Note that the above argument shows that

$$|Tf_n(x)| = \left| \int_0^x f_n(t)dt \right| \leq \sqrt{x}\|f_n\|_2 \leq \|f_n\|_2,$$

so the family is uniformly bounded. (The functions are continuous because f is locally integrable. In fact, you know from measure theory that they are absolutely continuous.)

To show that $\{Tf_n\}$ is an equicontinuous family, we take $\epsilon > 0$, and let $\delta = \frac{\epsilon^2}{C^2}$, where $\|f_n\|_2 \leq C$ for all n. Then if $|x-y| < \delta$, we have

$$|Tf_n(x) - Tf_n(y)| = \left| \int_x^y f_n(t)dt \right| \leq \sqrt{x-y}\|f_n\|_2 \leq \epsilon$$

by Cauchy-Schwarz. Thus the family is equicontinuous, and so we may apply Arzela-Ascoli.

2. Suppose that $\lambda \in \mathbb{C}$, and that $f \in L^2([0,1])$ is such that $Tf = \lambda f$.

   If $\lambda \neq 0$, then $Tf = \lambda f$ implies that f is absolutely continuous and is differentiable almost everywhere (more precisely, it has a representative that is). Taking the derivative, then, we see that $f'(x) = \frac{1}{\lambda} f(x)$ (almost everywhere) must also be absolutely continuous and differentiable almost everywhere. Thus f must be a continuously differentiable function that satisfies $f'(x) = \frac{1}{\lambda} f(x)$, so $f(x) = Ce^{x/\lambda}$. This does not satisfy $Tf = f$ unless $f = 0$, so we must have $f = 0$.

   If $\lambda = 0$, then we must have $\int_0^x f(t)dt = 0$ for all x. We know from measure theory that this implies that $f(t) = 0$ almost everywhere and hence $f = 0$ in L^2.

3. Let's first calculate

$$\begin{aligned} \left|T^2 f(x)\right| &= \left| \int_0^x \int_0^y f(t)dtdy \right| \leq \int_0^x \int_0^y |f(t)|dtdy \\ &\leq \int_0^x \sqrt{y}\|f\|_2 dy \leq \frac{2}{3} x^{3/2} \|f\|_2 \leq \frac{2}{3}\|f\|_2. \end{aligned}$$

   If we square and integrate, we find that $\|T^2\| \leq \frac{2}{3}$. Note that this immediately gives us that

$$\lim_{n\to\infty} \|T^n\| = 0,$$

   as $\|T^{2k}\| \leq \|T^2\|^k$, and $\|T^{2k+1}\| \leq \|T\|\|T^{2k}\|$.

   In order to conclude that T has spectrum $\{0\}$ using this line of reasoning, though, we need a somewhat finer estimate. Let's calculate $|T^n f(x)|$. Suppose, for induction, that

$$|T^k f(x)| \leq \left(\prod_{j=1}^{k-1} \frac{2}{2j+1} \right) \|f\|_2 x^{(2k+1)/2}.$$

   Then we have

$$\begin{aligned} \left|T^{k+1} f(x)\right| &= \left| \int_0^x T^k f(y)dy \right| \leq \int_0^x |T^k f(y)|dy \\ &\leq \int_0^x \left(\prod_{j=1}^{k-1} \frac{2}{2j+1} \right) \|f\|_2 y^{(2k+1)/2} dy = \left(\prod_{j=1}^{k} \frac{2}{2j+1} \right) \|f\|_2 y^{(2k+3)/2}, \end{aligned}$$

so this estimate holds for all k.

Squaring and integrating, we then see that $\|T^k\| \leq \prod_{j=1}^{k-1} \frac{2}{2j+1}$. Note that this proves that

$$\lim_{n\to\infty} \|T^n\| = 0.$$

The spectral radius formula tells us that the spectral radius,

$$r(T) = \sup_{\lambda\in\sigma(T)} |\lambda|,$$

is equal to the limit

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

Here we have that

$$\|T^n\|^{1/n} \leq \left(\prod_{j=1}^{n-1} \frac{2}{2j+1}\right)^{1/n},$$

which we can make as small as we like. Indeed, fix $\epsilon > 0$. Let k be such that $\frac{2}{2k+1} < \epsilon$. Then

$$\left(\prod_{j=1}^{n-1} \frac{2}{2j+1}\right)^{1/n} = \left(\prod_{j=1}^{k} \frac{2}{2j+1}\right)^{1/n} \left(\left(\prod_{j=k+1}^{n-1} \frac{2}{2j+1}\right)^{1/(n-2-k)}\right)^{(n-2-k)/n}.$$

The first of the two factors is bounded by 1, while for large n, the second of the two factors is bounded by a constant times $\frac{2}{2k+1}$, so the limit is zero. This proves that the spectrum of T is contained in $\{0\}$. T is compact, and so cannot be invertible, 0 is in the spectrum of T, finishing the proof.

Another way to approach this problem is to suppose that $\lambda \in \mathbb{C}$, $\lambda \neq 0$. (We already know that 0 is in the spectrum because T is compact.) We know from the second part of the problem that $T - \lambda I$ is injective, so if we can show it is surjective, then the open mapping theorem tells us that it must be an isomorphism. The surjectivity of $T - \lambda I$ is equivalent to the injectivity of $T^* - \overline{\lambda}$. Note that

$$\langle Tf, g\rangle = \int_0^1 \int_0^x f(t)dt\overline{g(x)}dx = \int_0^1 \int_x^1 \overline{g(t)}dtf(x)dx,$$

so that $T^*g(x) = \int_x^1 g(t)dt$. An identical argument to above then gives us that $T^* - \bar{\lambda}I$ is injective, and so $T - \lambda I$ is an isomorphism (meaning that λ is not in the spectrum).

Perhaps the simplest way to approach this problem is to use the Fredholm alternative. T is compact, so we know by the Fredholm alternative that for each $\lambda \neq 0$, $T - \lambda I = \lambda(\frac{1}{\lambda}T - I)$ is invertible if and only if it is injective. We know from the second part of the problem that for $\lambda \neq 0$, $T - \lambda I$ is injective and so λ is not in $\sigma(T)$. T is compact, so we already know that 0 is in the spectrum, and thus $\sigma(T) = \{0\}$.

■

II.1 Let $C([0,1])$ denote the Banach space of real-valued continuous functions on $[0,1]$ with the sup norm, and suppose that $X \subset C([0,1])$ is a dense linear subspace. Suppose $\ell : X \to \mathbb{R}$ is a linear map (not assumed to be continuous in any sense) such that $\ell(f) \geq 0$ if $f \in X$ and $f \geq 0$. Show that there is a unique Borel measure μ on $[0,1]$ such that $\ell(f) = \int f d\mu$ for all $f \in X$.

It is helpful in this problem that we are working with real-valued functions. X is dense, so by approximating the constant function 2, we know we may find $g \in X$ such that $g(x) \geq 1$ for all $x \in [0,1]$. Let $C = \ell(g)$.

Let $f \in X$, and let $h = \|f\|g - f$. $h \in X$ because X is linear, and $h \geq 0$ because $g \geq 1$. Thus $\ell(h) \geq 0$, and so

$$\ell(f) \leq \ell(g)\|f\| = C\|f\|,$$

so ℓ is bounded on X. X is dense, so ℓ extends uniquely to a bounded linear functional on all of $C([0,1])$. We know that the dual of $C([0,1])$ is the space of Borel measures on $[0,1]$, so there is a unique Borel measure μ representing ℓ, i.e.

$$\ell(f) = \int f d\mu$$

for all $f \in X$. ■

II.2 Prove that in the Banach space $C([0,1])$, the C^1 functions form a set of the first category.

First observe that

$$C^1([0,1]) \subseteq \cup_N \left\{ f \in C^1([0,1]) : \|f\|_{C^1} \leq N \right\} = \cup_N B_N.$$

We now claim that B_N is nowhere dense. This is the same as showing that, given $f \in \overline{B_N}$ and $\epsilon > 0$, we may find $g \notin \overline{B_N}$ such that $\|f - g\| \leq \epsilon$.

Note that if $f \in \overline{B_N}$, then for all $x, y \in [0,1]$,

$$|f(x) - f(y)| \leq N|x - y|.$$

The basic idea here is then to add a small but fierce sawtooth function to f.

Let

$$h(x) = \begin{cases} 2x & x \in [0,1/2] \\ -2(x-1) & x \in [1/2,1] \end{cases}$$

extended to be periodic with period 1. For $j \in \mathbb{N}$ and $\epsilon > 0$, let

$$h_{j,\epsilon}(x) = \epsilon h(2jx).$$

Note that $\|h_{j,\epsilon}\| \leq \epsilon$ and that

$$|h_{j,\epsilon}(1/2j) - h_{j,\epsilon}(0)| = \epsilon = (2j\epsilon) \left| \frac{1}{2j} - 0 \right|,$$

so that by choosing j large, we have

$$|h_{j,\epsilon}(x) - h_{j,\epsilon}(y)| > 3N|x-y|$$

for some $x, y \in [0,1]$.

Now, given such an $f \in \overline{B_N}$ and an $\epsilon > 0$, we take $h_{j,\epsilon}$ as above and let $g = h_{j,\epsilon} + f$. Then $\|g - f\| \leq \epsilon$ by construction, while

$$|g(1/j) - g(0)| \geq |h_{j,\epsilon}(1/j) - h_{j,\epsilon}(0)| + |f(1/j) - f(0)| \geq (2j\epsilon - N)\left|\frac{1}{j} - 0\right| \geq 2N\left|\frac{1}{j} - 0\right|,$$

so that g cannot be in the closure of B_N, and thus B_N is nowhere dense (so C^1 is a set of the first category in $C^0([0,1])$). ■

II.4 Let H be a separable infinite dimensional Hilbert space, and suppose that $e_1, e_2, \ldots$, resp. $f_1, f_2, \ldots$ are orthonormal systems in H. Assume that $\{\lambda_n, n \in \mathbb{N}\}$ is a bounded sequence of complex numbers, and let

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle f_n.$$

Show that

1. T is a bounded linear operator on H and $\|T\| = \sup |\lambda_n|$.
2. T is compact if and only if $\lambda_n \to 0$ as $n \to \infty$.
3. If K is a compact operator on H then there exist orthonormal systems $\{e_n\}$ and $\{f_n\}$, and a sequence $\{\lambda_n\}$ of complex numbers converging to 0 such that

$$Kx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle f_n$$

for all $x \in H$.

1. Let $A = \sup_n |\lambda_n|$. We may calculate, using Bessel's inequality,

$$\begin{aligned}\|Tx\|^2 &= \|\sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle f_n\|^2 \\ &= \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle x, e_n \rangle|^2 \\ &\leq A^2 \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq A^2\|x\|^2,\end{aligned}$$

so that T is bounded with $\|T\| \leq A$. Now let $\lambda_{n(j)}$ be a subsequence so that $|\lambda_{n(j)}| \to A$. Then we have

$$\|Te_{n(j)}\|^2 = |\lambda_{n(j)}|^2 \|f_n\|^2 = |\lambda_{n(j)}|^2 \to A^2,$$

so that $\|T\| \geq A$. Thus $\|T\| = A$.

2. Compact operators take weakly convergent sequences to norm convergent ones, so by considering the sequence e_n (which converges weakly to 0), we must have $|\lambda_n| = \|Te_n\| \to 0$. This proves one direction.

   Now suppose that $\lambda_n \to 0$. Consider the finite rank operators

$$T_k x = \sum_{n=1}^{k} \lambda_n \langle x, e_n \rangle f_n.$$

   By writing down an expression for $T - T_k$,

$$(T - T_k)x = \sum_{n=k+1}^{\infty} \lambda_n \langle x, e_n \rangle f_n,$$

   we may conclude by the first part that

$$\|T - T_k\| \leq \sup_{n \geq k+1} |\lambda_n|.$$

   We know that $\lambda_n \to 0$, so $\|T - T_k\| \to 0$. Thus T is a norm limit of finite rank operators, so we know that T must be compact.

3. The idea here is to try to use the spectral theorem for compact self-adjoint operators, which we know. Sadly, K may not be self-adjoint, so we consider the operator $A = K^*K$. Note that A is a compact operator because K is compact, that A is self-adjoint, and that A is positive:

$$\langle Ax, x \rangle = \langle K^*Kx, x \rangle = \langle Kx, Kx \rangle \geq 0.$$

   The spectral theorem for compact self-adjoint operators tells us that there is a sequence of eigenvalues $|\mu_n|^2 \downarrow 0$ of A ($|\mu_n|^2 \geq 0$ because A is positive) along with a complete orthonormal sequence of eigenvectors ϕ_n of A.

   For those n such that $\mu_n \neq 0$ (if $\mu_n = 0$ then $K\phi_n = 0$), we define

$$\psi_n = \frac{1}{\mu_n} K\phi_n.$$

   Note then that

$$\langle \psi_n, \psi_m \rangle = \frac{1}{\mu_n \overline{\mu_m}} \langle K\phi_n, K\phi_m \rangle = \frac{1}{\mu_n \overline{\mu_m}} \langle K^*K\phi_n, \phi_m \rangle = \frac{|\mu_n|^2}{\mu_n \overline{\mu_m}} \langle \phi_n, \phi_m \rangle = \delta_{nm},$$

   so that the ψ_n are orthonormal.

   Now, given any $x \in H$, we have that

$$\begin{aligned} Kx &= \sum_{n=1}^{\infty} \langle x, \phi_n \rangle K\phi_n \\ &= \sum_{n=1}^{\infty} \mu_n \langle x, \phi_n \rangle \psi_n, \end{aligned}$$

   and the $\mu_n \to 0$.

■

II.5 For $s \geq 0$, let $H^s(\mathbb{T})$ be the space of L^2 functions f on the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ whose Fourier coefficients $\hat{f}(n) = \int_0^{2\pi} e^{-inx} f(x) dx$ satisfy $\sum(1+n^2)^s |\hat{f}(n)|^2 < \infty$, with norm $\|f\|_s^2 = (2\pi)^{-1} \sum(1+n^2)^s |\hat{f}(n)|^2$.

1. Show that for $r > s \geq 0$, the inclusion map $H^r(\mathbb{T}) \hookrightarrow H^s(\mathbb{T})$ is compact.
2. Show that if $s > 1/2$, then $H^s(\mathbb{T})$ includes continuously into $C(\mathbb{T})$, and indeed the inclusion map is compact.

Let's start by introducing a different normalization in this problem. Let's say that $\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} f(x) dx$ and $\|f\|_s^2 = \sum(1+n^2)^s |\hat{f}(n)|^2$. This is so that $\widehat{e^{inx}}(n) = 1$ rather than 2π and that $\|e^{inx}\|_0 = 1$.

1. Call the inclusion map $J : H^r(\mathbb{T}) \hookrightarrow H^s(\mathbb{T})$, where $r > s \geq 0$. Consider the partial sum operators $J_N : H^r(\mathbb{T}) \to H^s(\mathbb{T})$, where

$$J_N f(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}.$$

We claim that $\|J - J_N\| \to 0$. Indeed, we have that for $f \in H^r(\mathbb{T})$,

$$\begin{aligned} \|(J - J_N)f\|_s^2 &= \sum_{|n|>N} (1+n^2)^s |\hat{f}(n)|^2 = \sum_{|n|>N} (1+n^2)^{s-r} (1+n^2)^r |\hat{f}(n)|^2 \\ &\leq (1+N^2)^{s-r} \sum (1+n^2)^r |\hat{f}(n)|^2 = (1+N^2)^{s-r} \|f\|_r. \end{aligned}$$

because $s - r < 0$. Thus $\|J - J_N\| \leq (1+N^2)^{s-r}$. $s - r < 0$, so $(1+N^2)^{s-r} \to 0$, so that J is the norm limit of finite rank operators and so is compact.

2. Now suppose that $s > 1/2$, and let $f \in H^s(\mathbb{T})$. Consider the sum

$$\begin{aligned} \left| \sum \hat{f}(n) e^{inx} \right| &\leq \sum |\hat{f}(n)| \\ &\leq \left(\sum (1+n^2)^{-s} \right)^{1/2} \left(\sum (1+n^2)^s |\hat{f}(n)|^2 \right)^{1/2} \\ &= \left(\sum (1+n^2)^{-s} \right)^{1/2} \|f\|_s \end{aligned}$$

by Cauchy-Schwarz. $s > 1/2$, so $(1+n^2)^{-s}$ is a summable sequence, and so this sum converges uniformly. Each summand is a continuous function, so the partial sums of $\sum \hat{f}(n) e^{inx}$ converge uniformly to a continuous function. This function has Fourier coefficients $\hat{f}(n)$, so by uniqueness, it must be f. Thus f is continuous. Note that the above calculation also shows that

$$\|f\|_{C^0} \leq \left(\sum (1+n^2)^{-s} \right)^{1/2} \|f\|_s,$$

so the inclusion is continuous.

∎