Dean Baskin

Qual Problems, Spring 2005

Spring 2005 Qual, Part I: 2, 3; Part II: 6, 7, 10

I.2 Prove that a linear operator T on a Hilbert space $\mathcal{H}$ is compact if, and only if, it is the limit in the operator norm topology of a sequence of operators of finite rank.

Suppose first that T is compact, i.e. the image of a bounded set is precompact. In particular, the closure of the image of the unit ball in $\mathcal{H}$ is compact. Recall the characterization of compact sets as sets that are complete and totally bounded.

Fix $\epsilon > 0$. Suppose that B is the unit ball in $\mathcal{H}$. T is compact, so TB is totally bounded, i.e. there are $y_1, y_2, \ldots, y_N$ such that for all $x \in B$, there is some i such that $\|Tx - y_i\| < \epsilon/2$. Without loss of generality, we may assume that the y_i are in the image of T. Let P be the orthogonal projctor onto the span of the y_i. Note that PT is a finite rank operator and that $\|Pz - z\| \leq \|y - z\|$ for all y in the span of the y_i. If $\|x\| = 1$, we then have that

$$\|PTx - Tx\| \leq \|y_i - Tx\| \leq \epsilon = \epsilon\|x\|.$$

$PT - T$ is linear, so this estimate shows that $\|PT - T\| \leq \epsilon$ and so T can be approximated in norm by finite rank operators.

Now suppose that T can be approximated in norm by finite rank operators. Recall that finite rank operators are compact. (This is essentially because the unit ball in a finite dimensional vector space is compact.) We will show that if T is the norm limit of compact operators T_n then T must be compact (i.e. compact operators are closed in the norm topology).

Suppose that x_n is a bounded sequence (say $\|x_n\| < C$) and that $T_n \to T$ in norm, T_n compact. T_1 is compact, so there is a subsequence $x_{1,n}$ of x_n such that $T_1 x_{1,n}$ converges so some y_1. We then inductively choose a subsequence $x_{k+1,n}$ of $x_{k,n}$ such that $T_{k+1}x_{k+1,n}$ converges to y_{k+1}. If we let $z_k = x_{k,k}$, then z_k is a subsequence of the original sequence x_n such that $T_j z_k \to y_j$ for each j. (This is a standard diagonal construction.)

Now we use the norm convergence of the T_j. Let $\epsilon > 0$ and let j be such that $\|T - T_j\| < \frac{\epsilon}{3C}$. $T_j x_n$ converges, so we may take N large enough so that if $n, m > N$ then $\|T_j x_n - T_j x_m\| < \epsilon/3$. We then have that

$$\|Tx_n - Tx_m\| \leq \|T - T_j\| \|x_n\| + \|T_j x_n - T_j x_m\| + \|T - T_j\| \|x_m\| \leq \epsilon.$$

Thus Tx_n is a Cauchy (and so convergent) sequence. Thus T is compact. ■

I.3 Suppose B is a Banach space and $K \subset B$ is a subset. Recall that its convex hull, $ch(K)$, is the smallest convex subset of B which contains K.

1. Prove that if K is compact, then the closure of $ch(K)$ is compact as well.
2. Show that the set of indicator functions $\{\mathbb{1}_{[\tau,\tau+\frac{1}{5}]}(t) : \tau \in \mathbb{T}\}$ is compact in $L^1(\mathbb{T})$, and its convex hull is not.

1. Recall that a set is compact if it is complete and totally bounded. B is complete, and the closure of $ch(K)$ is closed, so it is enough to show that $ch(K)$ is totally bounded, i.e. for all $\epsilon > 0$, we may find $x_1, \dots, x_N$ such that for each $x \in ch(K)$, there is some i with $\|x - x_i\| < \epsilon$.

   Fix $\epsilon > 0$. K is compact, so it is totally bounded, i.e. there are $x_1, \dots, x_N \in K$ so that

   $$K \subset \cup_{i=1}^{N} B(x_i, \frac{\epsilon}{2}) = \{x_1, \dots, x_N\} + B(0, \frac{\epsilon}{2}) = X + B(0, \frac{\epsilon}{2}),$$

   where $X = \{x_1, \dots, x_N\}$. Note that $ch(X)$ is compact because it is the image of the compact set

   $$\{(\alpha_1, \dots, \alpha_N) \in [0,1]^N : \sum \alpha_i = 1\}$$

   under the continuous map

   $$(\alpha_1, \dots, \alpha_N) \mapsto \sum \alpha_i x_i.$$

   $ch(X)$ is compact, so it may be covered by finitely many balls of radius $\epsilon/2$, so there are $y_1, \dots, y_K \in ch(X)$ such that

   $$ch(X) \subset \cup_{i=1}^{K} B(y_i, \frac{\epsilon}{2}) = \{y_1, \dots, y_K\} + B(0, \frac{\epsilon}{2}).$$

   We know that $K \subset X + B(0, \epsilon/2)$, so it is easy to see that $ch(K) \subset ch(X) + B(0, \epsilon/2)$. We then have that

   $$ch(K) \subset ch(X) + B(0, \frac{\epsilon}{2}) \subset \{y_1, \dots, y_K\} + B(0, \frac{\epsilon}{2}) + B(0, \frac{\epsilon}{2}) = \{y_1, \dots, y_K\} + B(0, \epsilon),$$

   i.e. $ch(K)$ may be covered by finitely many balls of radius ϵ. We have thus shown that $ch(K)$ is totally bounded, and so its closure is compact.

2. Consider the map $\iota : \mathbb{S}^1 \to L^1(\mathbb{T})$ given by

   $$\tau \mapsto \mathbb{1}_{[\tau, \tau + \frac{1}{5}]}(t).$$

   If $f(t) = \mathbb{1}_{[0, \frac{1}{5}]}(t)$, then $\iota(\tau) = f(t - \tau)$. We know that such translations are continuous on L^1 (from measure theory) and so the map ι is continuous. $\mathbb{T}$ is compact, so the set of indicator functions

   $$\{\mathbb{1}_{[\tau, \tau + \frac{1}{5}]}(t) : \tau \in \mathbb{T}\}$$

   is compact in $L^1(\mathbb{T})$.

   Now consider the sequence $\tau_n = \frac{n}{5} (\bmod 2\pi)$. Let

   $$f_N(t) = \frac{1}{N+1} \sum_{n=0}^{N} \mathbb{1}_{[\tau_n, \tau_n + \frac{1}{5}]}(t).$$

   Note that f_N is in the convex hull of the above indicator functions, and that f_N converges to the constant function $\frac{1}{10\pi}$ in L^1. This constant function, however, is not in the convex hull of these indicator functions because $\frac{2\pi}{5}$ is not rational.

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II.6 (Quickies)

1. A distribution μ on $\mathbb{T}$ is *positive* if $\langle f, \mu \rangle \geq 0$ for every nonnegative $f \in C^\infty(\mathbb{T})$. Show that a positive distribution is a measure.

2. Assume $f_n \in L^2([0,1])$, $n \in \mathbb{N}$, and $\|f_n\| \leq 1$. Prove that $\mu(\{x : |f_n(x)| > n^{\frac{2}{3}}) < n^{-\frac{4}{3}}$, and conclude that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that the measure of the set $\{x : |f_n(x)| \leq n^{\frac{2}{3}}$ for $n > N\}$ exceeds $1 - \epsilon$.

1. Recall that the dual of continuous functions on $\mathbb{T}$ are the finite measures, so it is enough to show that μ is actually in the dual of the continuous functions, i.e. there is some C such that
$$|\langle f, \mu \rangle| \leq C\|f\|_{C^0}$$
for all $f \in C^\infty(\mathbb{T})$. (Smooth functions are dense in the continuous functions, so it is enough to show the inequality for smooth functions.)

   First suppose that f is real-valued. Let M be such that $\|f\|_{C^0} \leq M$, i.e. $M \pm f \geq 0$. Our assumption about μ tells us that $\langle M \pm f, \mu \rangle \geq 0$, i.e.
$$|\langle f, \mu \rangle| \leq M \, |\langle 1, \mu \rangle| \, ,$$
and so, letting $C = \langle 1, \mu \rangle$, we have
$$|\langle f, \mu \rangle| \leq C\|f\|_{C^0}.$$
This shows the claim for real-valued f. If f is complex-valued, we write $f = f_1 + if_2$, where f_1, f_2 are smooth and real-valued. Note that $\|f_i\|_{c^0} \leq \|f\|_{C^0}$, $i = 1, 2$. In this case we have that
$$|\langle f, \mu \rangle| \leq |\langle f_1, \mu \rangle| + |\langle f_2, \mu \rangle| \leq C(\|f_1\|_{C^0} + \|f_2\|_{C^0}) \leq 2C\|f\|.$$

2. We start by observing that
$$\begin{aligned} \mu(\{x : |f_n(x)| > n^{\frac{2}{3}}\}) &= n^{-\frac{4}{3}} \int_{\{x:|f_n(x)|>n^{\frac{2}{3}}\}} n^{\frac{4}{3}} \\ &\leq n^{-\frac{4}{3}} \int_{\{x:|f_n(x)|>n^{\frac{4}{3}}\}} |f_n(x)|^2 \\ &\leq n^{-\frac{4}{3}} \int |f_n(x)|^2 \\ &= n^{-\frac{4}{3}} \|f_n\|_{L^2}^2 \leq n^{-\frac{4}{3}}. \end{aligned}$$
Fix $\epsilon > 0$. Notice that $\sum n^{-\frac{4}{3}} < \infty$, so we may take N such that
$$\sum_{n>N} n^{-\frac{4}{3}} < \epsilon.$$

In this case we have that

$$\begin{aligned}\mu\left(\{x : |f_n(x)| \leq n^{\frac{2}{3}} \text{ for } n > N\}\right) &= 1 - \mu\left(\{x : |f_n(x)| > n^{\frac{2}{3}} \text{ for some } n > N\}\right)\\ &\geq 1 - \sum_{n>N} \mu\left(\{x : |f_n(x)| > n^{\frac{2}{3}}\}\right)\\ &\geq 1 - \sum_{n>N} n^{-\frac{4}{3}} \geq 1 - \epsilon.\end{aligned}$$

■

II.7 Let $\{f_n\}$ be an orthonormal sequence in $L^2([0,1])$. Prove that

$$S_n = \frac{1}{n}\sum_{m=1}^{n} f_m \to 0 \text{ a.e.}$$

First observe that $\|S_n\|_{L^2}^2 = \frac{1}{n}$. Indeed, we can use the orthonormality of the f_n to see

$$\begin{aligned}\|S_n\|_{L^2}^2 &= \frac{1}{n^2}\sum_{i,j=1}^{n} \langle f_i, f_j\rangle\\ &= \frac{1}{n^2}\sum_{i=1}^{n} 1 = \frac{1}{n}.\end{aligned}$$

We then know that if $\sum \lambda_j^{-1} < \infty$ then

$$\|\sum |S_{\lambda_j}|^2\|_{L^1} \leq \sum \|S_{\lambda_j}\|_{L^2}^2 = \sum \lambda_j^{-1} < \infty.$$

Thus the sum $\sum |S_{\lambda_j}|^2 \in L^1$ and so it must converge almost everywhere. In particular, we may conclue that $S_{\lambda_j} \to 0$ almost everywhere.

Now note that

$$\lambda_j = [j \log^2 j]$$

is such a sequence by the integral test. Indeed, $\sum [j\log^2 j]^{-1}$ converges if and only if $\int_K^\infty \frac{1}{x\log^2 x}dx < \infty$. The change of variables $y = \log x$ gives us that

$$\int_K^\infty \frac{1}{x\log^2 x}dx = \int_K^\infty \frac{1}{y^2}dy < \infty,$$

so $\sum \lambda_j^{-1} < \infty$.

Now suppose that $N \in (\lambda_j, \lambda_{j+1})$. We write

$$\begin{aligned}S_N &= \frac{1}{N}\sum_{m=1}^{\lambda_j} f_m + \frac{1}{N}\sum_{m=\lambda_j+1}^{N} f_m\\ &= \frac{\lambda_j}{N}S_{\lambda_j} + \frac{1}{N}\sum_{m=\lambda_j+1}^{N} f_m.\end{aligned}$$

The first term converges to zero almost everywhere by the above, so we must only show that the second term converges to zero almost everywhere. Call this tern g_N. We have that

$$\|g_N\|_{L^2}^2 = \frac{1}{N^2}(N - \lambda_j - 1) \le \frac{1}{\lambda_j^2}(\lambda_{j+1} - \lambda_j).$$

and we wish to bound

$$\{x : g_N(x) \not\to 0\} = \bigcup_{r \in \mathbb{N}} \bigcap_{k=1}^{\infty} \bigcup_{n \ge k} \{x : |g_n(x)| > 1/r\}.$$

Note that $\cap_{k=1}^{\infty} \cup_{n\ge k} \{x : |g_n(x)| > 1/r\}$ has measure bounded by $\cup_{n \ge k}\{x : |g_n(x)| > 1/r\}$. We can write

$$\bigcup_{n=\lambda_k+1}^{\infty} \{x : |g_n(x)| > 1/r\} = \bigcup_{j=k}^{\infty} \bigcup_{\lambda_j+1}^{\lambda_{j+1}} \{x : |g_n(x)| > 1/r\}.$$

We can then bound each of these terms using Chebyshev's inequality:

$$\left| \bigcup_{n=\lambda_j+1}^{\lambda_{j+1}} \{x : |g_n(x)| > 1/r\} \right| \le (\lambda_{j+1} - \lambda_j) \cdot \frac{r^2(\lambda_{j+1} - \lambda_j)}{\lambda_j^2}.$$

For our sequence $\lambda_j = [j \log^2 j]$, the mean value theorem (applied to $x \log^2 x$) lets us bound

$$\begin{aligned} r^2 \sum \frac{(\lambda_{j+1} - \lambda_j)^2}{\lambda_j^2} &\le r^2 \sum \frac{(\log^2(j+1) - 2\log(j+1))^2}{j^2 \log^4(j+1)} \\ &\le r^2 \sum \frac{C}{j^2} < \infty \end{aligned}$$

so that the measure of $\cup_{n \ge k}\{x : |g_n(x)| > 1/r\} \to 0$ as $k \to \infty$ for any fixed r. Thus $g_N \to 0$ almost everywhere.

For your convenience, here is Chebyshev's inequality and its proof:

$$\begin{aligned} |\{x : |f(x)| > \alpha\}| &= \int_{\{x:|f(x)|>\alpha\}} 1 \\ &\le \frac{1}{\alpha^p} \int_{\{x:|f(x)|>\alpha\}} |f(x)|^p \\ &\le \frac{\|f\|_{L^p}^p}{\alpha^p}. \end{aligned}$$

■

II.10 Let B be a Banach space and S a linear map from B into $C([0,1])$, such that if $\{v_n\} \subset B$ and $\|v_n\|_B \to 0$ then $Sv_n(x) \to 0$ pointwise in $[0,1]$. Prove that S is bounded; in particular, the assumptions $\|v_n\|_B \to 0$ implies that $Sv_n(x) \to 0$ uniformly.

Your instinct here should be to use the closed graph theorem. Indeed, S is a linear map defined on all of B, so the closed graph theorem tells you that if it has a closed graph then it must be bounded.

Recall that to show the graph of S is closed, it is enough to show that if $v_n \to v$ in B and $Sv_n \to f \in C([0,1])$, then we must have that $f = Sv$. Suppose we have such a sequence $v_n \to v$ with $Sv_n \to f$ in $C([0,1])$ (i.e. uniformly). $v_n \to v$, so $\|v_n - v\|_B \to 0$. Our assumption about the map S tells us that $S(v_n - v)(x) \to 0$ pointwise in $[0,1]$. We assumed that $Sv_n - f$ converged to 0 uniformly (and so pointwise), which tells us that $f(x) = Sv(x)$ almost everywhere, i.e. $f = Sv$. This tells us that the graph of S is closed and so S must be bounded by the closed graph theorem. ■