Ph.D. Qualifying Examination, Real Analysis

Spring 2006, part I

Do all the problems.

1 Quickies

a. Let $f_n \in L^p([0,1])$ where $1 < p < \infty$. Suppose that $||f_n||_p \leq 1$ and moreover that $f_n(x) \to 0$ for a.e. x. Prove that $f_n \to 0$ weakly in L^p.

b. Let $v_1, \ldots, v_N$ be a finite sequence of unit vectors in a Hilbert space $\mathcal{H}$. Suppose that there exists a number $a \in (0,1)$ such that

$$\langle v_i, v_j \rangle \leq -a, \qquad \forall\, i \neq j.$$

Find an upper bound for N in terms of a.

c. Let $h \in L^2(S^1)$ and assume that $h(t) \neq 0$ for a.e. $t \in S^1$. Prove that the subspace

$$V = \{P(t)h(t) : P(t) \text{ a trigonometric polynomial}\} \subset L^2(S^1)$$

is dense.

2 Let $\{f_k\}$ be a sequence of real-valued functions defined on $[-1,1]$ such that

$$|f_k(x) - f_k(y)| \leq \sqrt{|x-y|} + \frac{1}{k}$$

for all $k \geq 0$ and $x, y \in [-1,1]$. Suppose also that each $f_k(0) = 0$. Prove that some subsequence of the f_k converges uniformly to a continuous function f on $[-1,1]$.

3

a. Construct a sequence $\{f_n\}$ of positive continuous functions on $\mathbb{R}$ such that $f_n(x)$ is bounded as $n \to \infty$ when $x \in \mathbb{Q}$, but $f_n(x)$ is unbounded for $x \in \mathbb{R} \setminus \mathbb{Q}$.

b. Prove that there is no sequence $\{g_n\}$ of positive continuous functions such that $g_n(x)$ is bounded when $x \in \mathbb{R} \setminus \mathbb{Q}$, but $g_n(x)$ is unbounded when $x \in \mathbb{Q}$.

4 Let $A \subset \mathbb{R}$ be a Lebesgue measurable set with $0 < \mu(A) < \infty$ (μ is Lebesgue measure). Let

$$d(x,r) = \frac{\mu(A \cap [a-r, a+r])}{2r}.$$

Prove that there is a point $x \in \mathbb{R}$ such that

$$0 < \liminf_{r \to 0} d(x,r) \leq \limsup_{r \to 0} d(x,r) < 1.$$

5 Let C be a closed convex set in a Hilbert space $\mathcal{H}$. Prove that C contains a unique element of minimal norm.

Ph.D. Qualifying Exam, Real Analysis

Spring 2006, part II

Do all the problems.

1 Let $f \in \mathcal{C}^0([\alpha,\beta])$, where $0<\alpha<\beta<1$. For each $n=1,2,\dots$, define

$$P_n(x) = \frac{\int_\alpha^\beta f(u)[1-(u-x)^2]^n\,du}{\int_{-1}^1 (1-u^2)^n\,du}.$$

Show that $P_n(x)$ is a polynomial of degree at most $2n$ and that for any closed subinterval $[a,b] \subset (\alpha,\beta)$, $P_n \to f$ uniformly.

2 Let W be any vector space, and suppose that $u, v_1, \dots, v_k$ be linear functionals on W. Endow W with the weakest topology such that the functionals $v_1,\dots,v_k$ are all continuous. Suppose that u is continuous in this topology. Prove that u is a linear combination of the v_j.

3

a. Let f be a measurable real-valued function on a finite measure space $(X,\mathcal{B},\mu)$. Define

$$m_n(f) = \mu\left(\{x : 2^n \le |f(x)| < 2^{n+1}\}\right),$$

for $n = 0,1,2,\dots$. Show how to estimate the L^p norm of f purely in terms of the quantities $\mu_n(f)$.

b. Let S be a bounded operator on $L^\infty(X,\mathcal{B},\mu)$ with norm 2. Assume that S is also bounded on $L^1(X,\mathcal{B},\mu)$, with $||Sf||_{L^1} \le C_1||f||_{L^1}$. Prove that there exists some $C_2>0$ such that

$$||Sf||_{L^2} \le C_2||f||_{L^2}.$$

Hint: Let $f_n(x) = f(x)$ if $|f(x)| < 2^n$, but $f_n(x)=0$ otherwise. Observe that

$$m_n(Sf) = m_n(S(f-f_{n-2})) \le 2^{-n}||S(f-f_{n-2})||_{L^1}.$$

4 Suppose that $f(x)$ is a continuous function on $\mathbb{R}$, and that in fact $x^m f^{(n)}(x) \in \mathcal{C}_0(\mathbb{R})$ (continuous functions which tend to zero as $x \to \pm\infty$) for $0 \le m,n \le 3$. Prove the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(x+2\pi n) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}.$$

5 Suppose that $f \in L^1([0,1])$. Prove that there are nondecreasing sequences of continuous functions, $\{\varphi_k\}_{k=1}^\infty$ and $\{\psi_k\}_{k=1}^\infty$, on $[0,1]$ such that for a.e. $x \in [0,1]$ (with respect to Lebesgue measure), both $\varphi_k(x)$ and $\psi_k(x)$ are bounded sequences, and moreover,

$$f(x) = \lim_{k\to\infty} \varphi(x) - \lim_{k\to\infty} \psi_k(x).$$