Ph.D. Qualifying Exam problems, Real Analysis

June 2005, part I.

1 (Quickies)

a. Let $(B, ||\cdot||)$ be a normed space, and $A : B \to B$ an invertible linear transformation such that $\|A^n\| \leq c$ for some constant $c > 0$ and all $n \in \mathbb{Z}$. Prove that there is an equivalent norm on B with respect to which A is an isometry.

ANSWER: For $x \in B$ set

$$\|x\|_* = \limsup \frac{1}{2N} \sum_{-N}^{N} \|A^n x\|.$$

b. Let μ be a finite measure on $[-1, 1]$ and assume that $\int x^{kn} = 0$ for some integer k and all nonnegative integers n. Prove that if k is odd then $\mu = 0$. What can you say when k is even?

ANSWER: If k is odd, the polynomials in x^k are dense in $C([-1, 1])$.

If k is even, the polynomials in x^k are dense in the subspace of even functions in $C([-1, 1])$, and hence, for all $f \in C([-1, 1])$,

$$\int (f(x) + f(-x))d\mu = \int f(x)d(\mu(x) + \mu(-x)) = 0$$

which is equivalent to $\mu(x) + \mu(-x) = 0$.

c. Prove that the space $C(0, 1)$ is not reflexive.

Hint: Identify the dual space $M(0, 1)$ of $C(0, 1)$ and show that the dual of $M(0, 1)$ contains elements that can not be identified with elements of $C(0, 1)$.

ANSWER: The dual space $M(0, 1)$ is the space of finite Borel measures (the Riesz representation theorem).

The map $\Phi : \ \mu \mapsto \mu([0, \frac{1}{2}])$ is in the dual space of $M(0, 1)$.

To see that there is no continuous function f such that $\langle \Phi, \mu \rangle = \inf f d\mu$, denote by δ_t the Dirac measure at t and observe that $\langle \Phi, \delta_t \rangle = \mathbb{1}_{[0,\frac{1}{2}](t)}$, while $\int f \, d\delta_t = f(t)$ is continuous.

d. Prove that $C(0, 1)$ is not isomorphic—and in particular not isometric— to a uniformly convex Banach space.

ANSWER: Uniformly convex Banach spaces are reflexive, and reflexivity is preserved under isomorphism. $C(0, 1)$ is not reflexive.

2 Prove that a linear operator T on a Hilbert space $\mathcal{H}$ is *compact* if, and only if, it is the limit, in the norm topology of operators, of a sequence of operators of finite rank.

ANSWER: By definition, T is compact if $TB(0,1)$, the T image of the unit ball $B(0,1)$, is precompact (has compact closure), or equivalently, totally bounded.

Assume T compact. Let $\varepsilon > 0$. Since $TB(0,1)$ is totally bounded cover it by a finite number of balls of radius ε, and let V_ε be the subspace spanned by the centers of these balls. Let $\boldsymbol{\pi}_V$ be the orthogonal projection of $\mathcal{H}$ onto V, Then $\boldsymbol{\pi}_V T$ is of finite rank and, since Tv is within ε from V for every $v \in B(0,1)$, we have $\|T - \boldsymbol{\pi}_V T\| \le \varepsilon$.

Conversely, assume that for some $\varepsilon > 0$, there exists an operator S of finite rank such that $\|T - S\| < \varepsilon/2$. Since $SB(0,1)$ is a bounded set in a finite dimensional space, it can be covered by a finite number of balls of radius $\varepsilon/2$, and the concentric balls of radius ε cover $TB(0,1)$. If such S can be found for every $\varepsilon > 0$ then $TB(0,1)$ is totally bounded.

3 Suppose B is a Banach space and $K \subset B$ is a subset. Recall that its convex hull, $\mathbf{ch}(K)$, is the smallest convex subset of B which contains K.

a. Prove that if K is compact, then the closure $\overline{\mathbf{ch}(K)}$ of $\mathbf{ch}(K)$ is compact as well.

ANSWER: Since $\overline{\mathbf{ch}(K)}$ is closed, it is enough to show that it is totally bounded. $\mathbf{ch}(K)$ is the set of all finite convex combinations of elements of K.

The assumption that K is compact guarantees that given $\varepsilon > 0$ there exists a finite subset $K_\varepsilon \subset K$ which is $\varepsilon/2$ dense in K. The convex combinations of elements of K_ε are $\varepsilon/2$ dense in $\mathbf{ch}(K)$. If $N > \frac{2\max_{v\in K}\|v\|}{\varepsilon}$, the the (finite) set of convex combinations of elements of K_ε with coefficients of the form $\frac{j}{N}$ is ε-dense in $\overline{\mathbf{ch}(K)}$.

b. Show that the set of indicator functions $A = \{\mathbb{1}_{[\tau,\tau+\frac{1}{5}]}(t) : \tau \in \mathbb{T}\}$ is compact in $L^1(\mathbb{T})$, and its convex hull is not.

ANSWER: A is the range of the continuos map $\tau \mapsto \mathbb{1}_{[\tau,\tau+\frac{1}{5}]}$ of the compact space $\mathbb{T}$ into $L^1(\mathbb{T})$. The convex hull of A contains no continuous functions. Its closure contains all the convolutions $\varphi * \mathbb{1}_{[0,\frac{1}{5}]}$ where φ is non-negative, continuous, and of integral 1. All these are continuous.

4 Let $A_j \subset [0,\, 1]$, for $j = 1, 2, \ldots, N$ be Lebesgue measurable, $\mu(A_j) \geq \frac{1}{2}$.

Let $0 < a < 1/2$ and denote $E_a = \{x : x \in A_j$ for more than aN values of $j\}$.

Prove that $\mu(E_a) \geq \frac{1-2a}{2(1-a)}$.

Show that the estimate $\mu(E_a) \geq \frac{1-2a}{2(1-a)}$ is "best possible" (if it is to apply to all N).

Hint: Consider $F = \sum_1^N \mathbb{1}_{A_j}$.

ANSWER: Write $F = \sum_1^N \mathbb{1}_{A_j}$. Then

$$\frac{1}{2}N \leq \int F(x)\,dx \leq \int_{E_a} F(x)\,dx + aN(1 - \mu(E_a) \leq N\mu(E_a) + aN(1 - \mu(E_a)) \qquad (1)$$

and $N(1-a)\mu(E_a) \geq N\frac{1-2a}{2}$.

To show that the estimate can not be improved, given $0 < a < 1/2$, define A_j so that the inequalities in (1) are close to equalities.

All the sets will contain $I_{a,\varepsilon} = [0, \frac{1-2a}{2(1-a)} + \varepsilon]$, where $\varepsilon = \varepsilon_N \to 0$ as $N \to \infty$—so that on $I_{a,\varepsilon}$ one has $F(x) = N$—while the sets $A_j \cap [\frac{1-2a}{2(1-a)} + \varepsilon, 1]$ are chosen in a way to have even covering—F is essentially a constant ($< aN$).

5 The Hausdorff-Young inequality on the line states that if $1 \leq p \leq 2$ and $1/p + 1/q = 1$, then

$$f \in L^p(\mathbb{R}) \quad \text{implies} \quad \|\hat{f}\|_{L^q(\hat{\mathbb{R}})} \leq \|f\|_{L^p(\mathbb{R})}. \tag{2}$$

Prove that the converse is true: (2) *implies* $1/p + 1/q = 1$ *and* $1 \leq p \leq 2$.
Hint: For the first claim use scaling ($f_{\lambda,s} = \lambda^s f(\lambda x)$ for appropriate s, and its effect on the Fourier transform). For the second claim, show that if $\varphi_\lambda(x) = e^{i\lambda x^2}\mathbb{1}_{[-1,1]}$, then as $\lambda \to \infty$ $\|\hat{\varphi}_\lambda\|_\infty \to 0$ while $\|\hat{\varphi}_\lambda\|_2$ remains constant.

ANSWER: The map $f \mapsto f_{\lambda,p^{-1}}$ is an isometry on $L^p(\mathbb{R})$.

Let $f \in L^p(\mathbb{R})$. The Fourier transform of $f_{\lambda,p^{-1}}$ is

$$\widehat{f_{\lambda,p^{-1}}}(\xi) = \lambda^{\frac{1}{p}-1} \int f(\lambda x) e^{i\frac{\xi}{\lambda}\lambda x} d\lambda x = \lambda^{\frac{1}{p}-1+\frac{1}{q}}\ \lambda^{\frac{1}{q}} \hat{f}(\frac{\xi}{\lambda}) = \lambda^{\frac{1}{p}-1+\frac{1}{q}} \widehat{f_{\lambda^{-1},q}}.$$

If $\frac{1}{p} - 1 + \frac{1}{q} \neq 0$ one can choose λ (close to 0 if $\frac{1}{p} - 1 + \frac{1}{q} < 0$, large if $\frac{1}{p} - 1 + \frac{1}{q} > 0$) to make $\lambda^{\frac{1}{p}-1+\frac{1}{q}}$ arbitrarily big.

For the second claim: van der Corput's lemma gives $\|\hat{\varphi}_\lambda\|_\infty = O^*\left(\lambda^{-\frac{1}{2}}\right)$. Since $\|\hat{\varphi}_\lambda\|_2$ remains constant, $\int |\hat{\varphi}_\lambda(\xi)|^q d\xi = O\left(\lambda^{-\frac{q-2}{2}}\right)$.

This shows that if $q > 2$ the map $f \mapsto \hat{f}$ of $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$ is not bounded below, and the inverse Fourier transform, which is essentially the Fourier transform, cannot be defined as a bounded operator $L^q(\mathbb{R}) \to L^p(\mathbb{R})$.

Ph.D. Qualifying Exam problems, Real Analysis

June 2005, part II.

6 (Quickies)

a. A distribution μ on $\mathbb{T}$ is *positive* if $\langle f, \mu \rangle \geq 0$ for every nonnegative $f \in C^\infty(\mathbb{T})$.
Show that a positive distribution is (defined by) a measure.
Hint: Positivity implies: for real-valued C^∞ functions f, $\langle f, \mu \rangle \leq \max f(t) \langle 1, \mu \rangle$.

ANSWER: Use the hint for both f and $-f$ and conclude that for real-valued $f \in C^\infty(\mathbb{T})$,

$$|\langle f, \mu \rangle| \leq K \|f(t)\|_\infty$$

with $K = \langle 1, \mu \rangle$. Since $C^\infty(\mathbb{T})$ is dense in $C(\mathbb{T})$, μ has a unique extension to a bounded linear functional on $C(\mathbb{T})$. By the Riesz representation theorem it is a measure on $\mathbb{T}$.

b. Assume $f_n \in L^2[0,1]$, $n \in \mathbb{N}$, and $\|f_n\| \leq 1$.
Prove that $\mu(\{x \,|\, |f_n(x)| > n^{\frac{2}{3}}\}) < n^{-\frac{4}{3}}$, and conclude that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that the measure of the set $A_N = \{x : |f_n(x)| \leq n^{\frac{2}{3}} \text{ for all } n > N\}$ exceeds $> 1 - \varepsilon$.

ANSWER: The first estimate is Chebishev's (or *weak type*) inequality. The second claim follows from

$$\mathbb{T} \setminus A_N = \cup_{n>N} \{x : |f_n(x)| > n^{\frac{2}{3}}\}$$

and its measure is bounded by $\sum_{n>N} n^{-\frac{4}{3}} \sim N^{-\frac{1}{3}}$.

7 Let $\{f_n\}$ be an orthonormal sequence in $L^2(0,1)$.
Prove that $S_n = \frac{1}{n} \sum f_m \to 0$ a.e.
Hints:

a. $\|S_n\|_{L^2} = \frac{1}{n}$. It follows that if $\sum \lambda_j^{-1} < \infty$, and in particular if $\lambda_j = [j \log^2 j]$, then $\sum |S_{\lambda_j}|^2$ converges a.e. and $S_{\lambda_j} \to 0$ a.e.

b. If $N \in (\lambda_j, \lambda_{j+1})$, then $S_N = \frac{\lambda_j}{N} S_{\lambda_j} + \frac{1}{N} \sum_{\lambda_j+1}^{N} f_m$. Use **6.b.** to estimate the last sum.

8 Let $B = \{(x,y,z) \colon r = \sqrt{x^2+y^2+z^2} \le 1\}$, the unit ball in $\mathbb{R}^3$, $G = \mathbb{1}_B$ its indicator function, and $g(x) = \int G(x,y,z)\,dy\,dz$.

a. Compute g.

b. Show that $\hat{g}(\xi) = O\left(|\xi|^{-3}\right)$

c. Prove $\hat{G}(\xi,\eta,\zeta) = O\left((\xi^2+\eta^2+\zeta^2)^{-\frac{3}{2}}\right)$.

d. Let $F_n(x,y,z) = \sin^2(nr)G(x,y,z)$, and $f_n(x) = \int F_n(x,y,z)dy\,dz$.
Prove: $f_n \to \frac{1}{2}g$ uniformly as $n \to \infty$,

ANSWER:

a. $g(x) = \pi(1-x^2)$ for $|x| \le 1$, zero elsewhere.

b. Use the fact: $g'' = -2\pi\mathbb{1}_{[-1,1]}$.

c. $\hat{G}$ is radial and $\hat{G} = \hat{g}$ on the real axis.

d. The simplest form of van der Corput gives rate $O\left(\frac{1}{n}\right)$.

9 Let $f(x)=\sum 10^{-n}\cos 10^{2n}x$

a. Prove that f satisfies the Hölder $\frac{1}{2}$ condition:

$$|f(x+h)-f(x)|\le \text{const }\cdot h^{\frac{1}{2}}.$$

ANSWER: For $m\in\mathbb{N}$ write

$$S_m(x)=\textstyle\sum_{n<m}10^{-n}\cos 10^{2n}x, \quad \text{and} \quad T_m(x)=\sum_{n>m}10^{-n}\cos 10^{2n}x.$$

Given h, let m be such that $h\sim 10^{-2m}$ and write

$$f(x)=S_m(x)+10^{-m}\cos 10^{2m}x+T_m(x). \tag{3}$$

Observe that $\|\frac{d}{dx}S_m\|_\infty\le 1.2\,10^{m-1}$ and $\|T_m\|_\infty\le 1.2\,10^{-m-1}$ so that

$$\begin{aligned}|f(x+h)-f(x)|\le |S_m(x+h)-S_m(x)|+&\\ 10^{-m}|\cos 10^{2m}(x+h)-\cos 10^{2m}x|+|T_m(x+h)-T_m(x)|&. \end{aligned}\tag{4}$$

We have $|S_m(x+h)-S_m(x)|\le|h|\|\frac{d}{dx}S_m\|_\infty\sim 10^{-m}\sim|h|^{\frac{1}{2}}$ and the other two terms are bounded uniformly by $10^{-m}\sim|h|^{\frac{1}{2}}$.

b. Prove that f is nowhere differentiable.

Hint: For every x and every n find points y_n and z_n such that

$|x-y_n|\sim 10^{-2n}$, $|x-z_n|\sim 10^{-2n}$, $\frac{f(x)-f(y_n)}{x-y_n}>10^{n-2}$, and $\frac{f(x)-f(z_n)}{x-z_n}<-10^{n-2}$.

ANSWER: Given x an n, if $\cos 10^{2n}x>0$ let y_n be the point closest to x on the left such that $\cos 10^{2n}y_n=-1$ and z_n the closest on the right satisfying the same condition. If $\cos 10^{2n}x\le 0$ define y_n and z_n as the closest neighbors on either side on which $\cos 10^{2n}y_n=\cos 10^{2n}z_n=1$.

Since $\cos 10^{2n}t$ is $2\pi 10^{-2n}$ periodic, $\frac{\pi}{2}10^{-2n}\le|x-y_n|\le 2\pi 10^{-2n}$ and similarly for z_n. Thus,

$$\frac{10^{-n}\cos 10^{2n}x-10^{-n}\cos 10^{2n}y_n}{x-y_n}\ge\frac{2}{\pi}10^{2n}10^{-n}=\frac{2}{\pi}10^{n}\ge 0.5\,10^{n}.$$

Now write again

$$f(x)=S_n(x)+10^{-n}\cos 10^{2n}x+T_n(x). \tag{5}$$

and observe that

$$\Big|\frac{f(x)-f(y_n)}{x-y_n}-10^{-n}\frac{\cos 10^{2n}x-\cos 10^{2n}y_n}{x-y_n}\Big|\le\Big\|\frac{d}{dx}S_n\Big\|_\infty+\frac{4}{\pi}10^{2n}\|T_n\|_\infty\le 0.3\;10^{n}$$

It follows that

$$\left|\frac{f(x)-f(y_n)}{x-y_n}\right| > 0.2\ 10^n. \tag{6}$$

We obtain the same estimate, with reversed sign, for z_n.

c. Can a Lipschitz function be nowhere differentiable? (Justify your answer by quoting relevant standard theorems.)

ANSWER: No! a Lipschitz function is of bounded variation and, by Lebesgue's theorem, is differentiable a.e.

10 Let B be a Banach space and S a linear map from B into $C([0,1])$, such that if $\{v_n\} \subset B$ and $\|v_n\|_B \to 0$ then $Sv_n(x) \to 0$ pointwise in $[0,1]$. Prove that S is bounded; in particular, the assumptions $\|v_n\|_B \to 0$ implies $Sv_n(x) \to 0$ *uniformly*.

ANSWER: For every $x \in [0,1]$ the map $\varphi_x : v \mapsto Sv(x)$ is a bounded linear functional on B. The problem is to show that these fuctionals are uniformly bounded.

Assume they are not. Let $\{x_n\}$ be such that $\|\varphi_{x_1}\| > 1$, and $\|\varphi_{x_n}\| > 10^n \|\varphi_{x_{n-1}}\|$ for all $n > 1$. Let $v_n \in B$ such that $\|v_n\| = 5^{-n} \|\varphi_{x_{n-1}}\|$, and $Sv_n(x_n) > 2^n$. Write $v = \sum v_n$. Since $S(v - \sum_{n=1}^m)v_n$ converges to zero pointwise, $Sv = \sum Sv_n$ which is not bounded.