Algebra Qualifying Exam, Spring 1998: Part I

Directions: Work each problem in a separate bluebook. Give reasons for your answers, and make clear which facts you are assuming. If you have any questions about notation, terminology the meaning of a problem or the level of detail appropriate, please do not hesitate to ask the proctor.

Notation:
$\mathbb{Z}$: Integers
$\mathbb{Q}$: Rational Field
$\mathbb{R}$: Real Field
$\mathbb{C}$: Complex Field
$GL(,)$: Full linear group
$\mathbb{F}_q$: Finite field with q elements

1. Let G be a group of order p^n where p is prime and $n > 1$. Show that G has an automorphism of order p.

2. Let M be a 9×9 matrix over $\mathbb{C}$ with characteristic polynomial $(x^2+1)^3(x+1)^3$ and with minimal polynomial $(x^2+1)^2(x+1)$.

(a) Find trace(M) and det(M).
(b) How many distinct conjugacy classes of such matrices are there in $GL(9, \mathbb{C})$? Write down the Jordan form over $\mathbb{C}$ for one such matrix M.
(c) Write down a 9×9 matrix with rational coefficients with the above characteristic and minimal polynomials.

3. Let $F = \mathbb{C}(z)$ be the field of rational functions in one variable over $\mathbb{C}$, i.e. the field of fractions of the polynomial ring $A = \mathbb{C}[z]$.

(a) Show that $A = \mathbb{C}[z]$ is integrally closed in $F = \mathbb{C}(z)$.
(b) Show that $f(y) = y^5 - (z+1)(z+2) \in F[y]$ is irreducible.
(c) Let $E = F[y]/(f(y))$ and let $B = \mathbb{C}[z, y] \subset E$ be the integral closure of $A = \mathbb{C}[z]$ in E. Consider the prime ideals $\mathfrak{p}_0 = (z)$ and $\mathfrak{p}_1 = (z+1)$ in A. How many prime ideals in B lie above $\mathfrak{p}_0$ and $\mathfrak{p}_1$, respectively.

4. Suppose G is a finite group, $N \subset G$ is a subgroup, and $\rho : G \to \mathrm{End}(V)$ is an irreducible complex representation of G. Suppose there is a nonzero vector $v_0 \in V$ such that $\rho(x)v_0 = v_0$ for all $x \in N$.

(a) If N is normal in G, prove that N is contained in the kernel of ρ.
(b) Give an example to show that the conclusion to (a) need not be true if N is not normal in G.

5(a). Suppose that F is a field of characteristic $p > 0$. If α is algebraic over F, show that α is separable over F if and only if $F(\alpha) = F(\alpha^{p^n})$ for all $n \geq 1$.

(b). Suppose that k is a field of characteristic $p > 0$ and let $F = k(x, y)$ by the field of rational functions in two independent variables over k. Let $E = F(x^{1/p}, y^{1/p})$. Prove that E is not primitively generated over F. In other words, prove for all $\theta \in E$ that $F(\theta) \neq E$.

Algebra Qualifying Exam, Fall 1998: Part II

Directions: Work each problem in a separate bluebook. Give reasons for your answers, and make clear which facts you are assuming. If you have any questions about notation, terminology the meaning of a problem or the level of detail appropriate, please do not hesitate to ask the proctor.

Notation:
$\mathbb{Z}$: Integers
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$\mathbb{C}$: Complex Field
$GL(\ )$: Full linear group
$\mathbb{F}_q$: Finite field with q elements

1. Classify groups of order 306 that have a cyclic 3-Sylow subgroup.

2(a). Find the order of $GL(5, \mathbb{F}_2)$, the group of invertible 5×5 matrices over the field $\mathbb{F}_2$.
(b). Show that the polynomial $f(x) = x^5 + x^3 + x^2 + x + 1 \in \mathbb{F}_2[x]$ is irreducible. (Hint: how many irreducible quadratics are there over $\mathbb{F}_2$?)
(c). Exhibit a matrix A of order 31 in $GL(5, \mathbb{F}_2)$. (Hint: use (b) and some finite field theory.)

3. Let R be a commutative ring and let E be an R-module spanned over R by elements $e_1, \ldots, e_n$. Suppose that $b : E \times E \to R$ is an R-bilinear map such that $\det(B) \in R$ is *not* a zero divisor, where B is the $n \times n$ matrix $(b(e_i, e_j))$. Prove that E is a free R-module.

4. Let G be the group of order $136 = 8 \cdot 17$ with presentation

$$\langle x, y : y^8 = x^{17} = 1, \quad yxy^{-1} = x^4 \rangle.$$

(a) Find the center of G.
(b) Describe the number and dimensions of the irreducible complex representations of G.
(c) Find the simple summands of the group ring $\mathbb{Q}[G]$.

5(a). Let ζ be a primitive 7th root of unity in $\mathbb{C}$ and let $\beta = \zeta + \zeta^2 + \zeta^4$. Show that $[\mathbb{Q}(\beta) : \mathbb{Q}] = 2$ and that $\sqrt{-7} \in \mathbb{Q}(\beta)$. (Hint: find a linear relation between 1, β, and β^2.)
(b). Let E be the splitting field of the polynomial $x^{14} + 7 = f(x)$ over $\mathbb{Q}$ and let α be a root of $f(x)$ in $\mathbb{C}$. Show that $E = \mathbb{Q}[\zeta, \alpha]$ and find the degrees $[E : \mathbb{Q}]$, $[E : \mathbb{Q}(\zeta)]$, and $[E : \mathbb{Q}(\alpha)]$.
(c). Write down elements σ and τ of orders 6 and 7, respectively, in $\mathrm{Gal}(E/\mathbb{Q})$ by explicitly giving the values $\sigma(\zeta)$, $\sigma(\alpha)$, and $\tau(\zeta)$, $\tau(\alpha)$.