FALL QUALIFYING EXAM 2010, ALGEBRA

(1) (a) Let G be a finite group of order n, and let $\rho : G \to \mathrm{Sym}(G)$ be the homomorphism that arises from G acting on itself by left translation. Let $g \in G$ have order m. Prove that the sign of $\rho(g)$ is $(-1)^{n+m}$.

(b) Let G be a finite group of order $2k$, with k odd. Prove that G is a semidirect product $N \ltimes (\mathbb{Z}/2\mathbb{Z})$, where N has order k. Hint: Using part (a), construct a nontrivial homomorphism $G \to \mathbb{Z}/2\mathbb{Z}$.

(2) Let G be a group, and k a field.

(i) Define the group algebra $k[G]$, and briefly explain why a k-linear representation of G is "the same" as a left $k[G]$-module.

(ii) Explain how to make $A \otimes_k B$ naturally into an associative k-algebra for any two associative k-algebras A and B, and construct a natural k-algebra isomorphism $k[G] \otimes_k k[H] \simeq k[G \times H]$ for any two groups G and H.

(iii) For an associative k-algebra A, define its *opposite* algebra A^{opp} to have underlying k-vector space A but the multiplication law $a \times a' := a'a$ ("flipped around"). Explain briefly why this is is an associative k-algebra, and for $A = k[G]$ prove that $A^{\mathrm{opp}} \simeq A$. (Hint: use inversion in G).

(3) Suppose A is a Noetherian domain. Show that A is a unique factorization domain if and only if every prime ideal minimal over (0) (i.e. those $\mathfrak{p} \neq (0)$ such that there is no prime $\mathfrak{q}$ strictly between $\mathfrak{p}$ and (0)) is principal.

(4) Let $f = X^3 - 2 \in \mathbb{Z}[X]$.

(a) Prove f is irreducible over $\mathbb{Q}$ and that its splitting field $K/\mathbb{Q}$ has Galois group S_3.

(b) For each subgroup H of S_3, determine with proof the corresponding field K^H.

(c) Prove that the splitting field of f over $\mathbb{F}_5$ is quadratic and over $\mathbb{F}_7$ is cubic.

(5) Let $G = GL(2, \mathbb{F}_q)$, where $q = p^n$, p prime. Let Π be the set of one-dimensional subspaces in $V = \mathbb{F}_q^2$. Since G acts on V by matrix multiplication, it acts on Π.

(a) Show that if $\ell \in \Pi$ then the stabilizer of ℓ in G contains a unique p-Sylow subgroup of G. How many p-Sylow subgroups does G have, and what is their order?

(b) Prove that if ℓ_1, ℓ_2 and ℓ_3 are three distinct one-dimensional subspaces of V, then there is an element g of G such that $g\ell_1 = \mathbb{F}_q \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $g\ell_2 = \mathbb{F}_q \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $g\ell_3 = \mathbb{F}_q \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(c) Show that if P_1, P_2 and P_3 are three distinct p-Sylow subgroups of G, and if Q_1, Q_2 and Q_3 are another three distinct p-Sylow subgroups of G, then there exists a $g \in G$ such that

$$gP_1g^{-1} = Q_1, \qquad gP_2g^{-1} = Q_2, \qquad gP_3g^{-3} = Q_3.$$

(1) Let G, H be finite p-groups.
   (i) Lead-in on derived series to be added.
   (ii) Let $f : G \to H$ be a homomorphism so that the induced homomorphism $G/[G,G] \to H/[H,H]$ is surjective. Prove that f is surjective.

(2) Let G be a finite group and $K \subset L$ an extension of fields of any characteristic. For a K-vector space W, let W_L denote $L \otimes_K W$. Let V, V' be n-dimensional K-linear representations.
   (i) Prove that there is a nonzero $K[G]$-linear map $V'_L \to V_L$ if there is a nonzero $L[G]$-linear map $V' \to V$.
   (ii) Prove that a nonzero polynomial over K in N variables cannot vanish on K^N if K is infinite, and deduce that, if K is infinite,

$$V'_L \simeq V_L \text{ as } L[G]\text{-modules} \implies V' \simeq V \text{ as } K[G]\text{-modules}.$$

(3) Let K be a field and L, L' two finite extensions of K.
   (a) Prove that if L/K is separable then $L \otimes_K L'$ is isomorphic to a product $\prod_{i=1}^r L_i$ of finitely many fields. (Hint: use the primitive element theorem to write $L = K[t]/(f)$ for suitable f.)
   (b) Give an example to show this need not be true without the separability assumption.

(4) We say an A-module M is *finitely presented* if it is isomorphic to the cokernel of a map $A^{\oplus m} \to A^{\oplus n}$.
   (i) Suppose B is a flat A-algebra, M is a finitely presented A-module, and N is *any* A-module. Show that the natural map $\mathrm{Hom}_A(M, N) \otimes_A B \to \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B)$ is an isomorphism.

   (ii) Suppose

$$(*) \quad 0 \to M' \to M \to M'' \to 0$$

   is a short exact sequence of A-modules. Suppose that for each maximal ideal $\mathfrak{m} \subset A$, the localized sequence

$$0 \to M'_{\mathfrak{m}} \to M_{\mathfrak{m}} \to M''_{\mathfrak{m}} \to 0$$

   is split. If M'' is finitely presented, show that $(*)$ is split.
   Hint: Show that $(*)$ is split if and only if the map $\mathrm{Hom}_A(M'', M) \to \mathrm{Hom}_A(M'', M'')$ induced by the right map $M \to M''$ of $(*)$ is surjective.

(5) Let F be a field, and let V be the 4-dimensional vector space F^4 with the skew-symmetric bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_3 + x_2 y_4 - x_3 y_1 - x_4 y_2.$$

Let us say that a subspace U of V is *isotropic* if $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$ for $\mathbf{u}_1, \mathbf{u}_2 \in U$. For example the two-dimensional subspace U_0 of $\mathbf{x} = (x_1, x_2, x_3, x_4)$ with $x_3 = x_4 = 0$

is isotropic. Let G be the group of $g \in GL_4(F)$ such that $\langle g\mathbf{x}, g\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$ for all $g \in G$.

(a) Prove that if U is a two-dimensional isotropic subspace then there exists $g \in G$ such that $gU = U_0$. **Hint:** let $\mathbf{u}_1, \mathbf{u}_2$ be any basis of U. Show that there exist $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0$ and $\langle \mathbf{u}_i, \mathbf{v}_j\rangle = \delta_{ij}$.

(b) Suppose that $F = \mathbb{F}_q$. Show that the number of isotropic subspaces of V is $(q^2 + 1)(q + 1)$.