ALGEBRA QUAL, SPRING 2009, PART I

1. (a) [5 points] Prove that if A is a commutative noetherian ring then the polynomial ring $A[T]$ is noetherian. (That is, prove the 'Hilbert Basis Theorem'.)

(b) [3 points] Suppose k is a field and B is a commutative ring finitely generated over k. Let $S \subset B$ be a multiplicative set. Explain why the localization $S^{-1}B$ is a noetherian ring.

(c) [2 points] Give an example (with proof) of the situation in part (b) where $S^{-1}B$ is *not* finitely generated over k as a ring.

2. Let k be a field, $f(x) \in k[x]$ a monic, non-constant polynomial.

(a) [2 points] Define what it means for a field $K \supset k$ to be a *splitting field* of $f(x)$ over k.

(b) [8 points] Prove the existence of such a splitting field K, and the uniqueness of K up to isomorphism over k.

3. Let K be a splitting field for $x^4 - 7$ over $\mathbb{Q}$.

(a) [5 points] Determine $|K : \mathbb{Q}|$ and give field generators for K over $\mathbb{Q}$. Describe $G = Gal(K/\mathbb{Q})$ in terms of generators and relations, and describe how the group generators of G act on the field generators of K.

(d) [5 points] List all intermediate fields L with $\mathbb{Q} \subsetneqq L \subsetneqq K$, their degrees over $\mathbb{Q}$, and the inclusion relations that hold between the fields L. The fields L should be named in terms of generators over $\mathbb{Q}$.

4. Consider the finite groups $SL(2, \mathbb{F}_5)$ and $PSL(2, \mathbb{F}_5) = SL(2, \mathbb{F}_5)/\{\pm Id\}$. In the following problem, you are not allowed to use the fact that $PSL(2, \mathbb{F}_5)$ is isomorphic to a more familiar group, unless you give a complete proof of that fact.

(a) [2 points] Calculate $|SL(2, \mathbb{F}_5)|$. Explain why $SL(2, \mathbb{F}_5) \not\cong S_5$, the symmetric group.

(b) [3 points] Show that there are no elements of order 15 in $PSL(2, \mathbb{F}_5)$.

[Hint: Work in $SL(2, \mathbb{F}_5)$.]

(c) [5 points] Exhibit a 3-Sylow subgroup and a 5-Sylow subgroup of $PSL(2, \mathbb{F}_5)$ and determine (with proof) the number of distinct 3-Sylow subgroups and 5-Sylow subgroups.

[Hint: Part (b) is useful for (c), but there are various approaches.]

5. Let $\mathbb{Q}[x_1, \cdots, x_k]$ be the polynomial ring in k variables over $\mathbb{Q}$, and let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$, say inside $\mathbb{C}$. A special case of the *weak Nullstellensatz* states that if $I \subset \mathbb{Q}[x_1, \cdots, x_k]$ is any proper ideal, then

$$V(I) = \{\vec{\gamma} = (\gamma_1, \cdots, \gamma_k) \in (\overline{\mathbb{Q}})^k \mid f(\vec{\gamma}) = 0 \text{ for all } f \in I\} \neq \varnothing.$$

(a) [3 points] Use the weak Nullstellensatz in the form stated above to prove the *strong Nullstellensatz*, in the form that for any proper ideal $J \subset \mathbb{Q}[x_1, \cdots, x_n]$ the radical $\sqrt{J}$ is given by

$$\sqrt{J} = \{g \in \mathbb{Q}[x_1, \cdots, x_n] \mid g(\vec{\gamma}) = 0 \text{ for all } \vec{\gamma} \in V(J) \subset (\overline{\mathbb{Q}})^n\}.$$

[Hint: Make use of k = n+1 in the weak Nullstellensatz.]

(b) [2 points] Explain why $\sqrt{J}$ is the intersection of all *maximal* ideals $Q \subset \mathbb{Q}[x_1, \cdots, x_n]$ with $J \subset Q$.

(c) [5 points] If $P \subset \mathbb{Q}[x_1, \cdots, x_n]$ is a *minimal nonzero* prime ideal, prove that $P = (f)$, where $f \in \mathbb{Q}[x_1, \cdots, x_n]$ is irreducible. Then prove that there is a j, $1 \leq j \leq n$, so that the $n-1$ elements $\{\bar{x}_i = x_i \mod P \mid i \neq j\}$ are algebraically independent over $\mathbb{Q}$ in the integral domain $\mathbb{Q}[x_1, \cdots, x_n]/P$.

ALGEBRA QUAL, SPRING 2009, PART II

1. (a) [2 points] Prove that every finite field $\mathbb{F}$ has order $q = p^n$ for some prime integer p and some integer $n \geq 1$.

(b) [5 points] Prove that for each such $q = p^n$ there is up to isomorphism exactly one field $\mathbb{F}_q$ of order q.

[You may use the existence and uniqueness of splitting fields of polynomials.]

(c) [3 points] Prove that $K = \mathbb{F}_3[x]/(x^2 + x - 1)$ and $K' = \mathbb{F}_3[y]/(y^2 + 1)$ are fields and exhibit an *explicit* isomorphism between them.

2. Suppose G_1 and G_2 are groups and $H \subset G_1 \times G_2$ is a subgroup so that the two compositions

$$\begin{array}{rcll} p_1 : H & \subset & G_1 \times G_2 & \rightarrow G_1 \\ p_2 : H & \subset & G_1 \times G_2 & \rightarrow G_2 \end{array}$$

are *surjections.* Let $N_1 = ker(p_2)$ and $N_2 = ker(p_1)$. Thus, if $e_1 \in G_1$ and $e_2 \in G_2$ are the identity elements then

$$\begin{array}{rclcl} N_1 & = & H \cap (G_1 \times \{e_2\}) & \subset & G_1 \times \{e_2\} \\ N_2 & = & H \cap (\{e_1\} \times G_2) & \subset & \{e_1\} \times G_2. \end{array}$$

(a) [5 points] Show that $N_1 \lhd G_1 \times \{e_2\}$ and $N_2 \lhd \{e_1\} \times G_2$ are normal subgroups.

(b) [5 points] Show that
$$\frac{G_1 \times \{e_2\}}{N_1} \simeq \frac{\{e_1\} \times G_2}{N_2}.$$

3. Let $T : V \rightarrow V$ be a linear endomorphism of a non-zero finite dimensional vector space over $\mathbb{C}$.

(a) [4 points] State precisely the theorem on the *existence* and *uniqueness* of a Jordan canonical form for T, and prove it using the structure theorem for modules over a PID.

(b) [2 points] Using the Jordan form, prove that $T = T_s + T_n$, where $T_s : V \rightarrow V$ is diagonalizable and $T_n : V \rightarrow V$ is nilpotent, and where $T_sT_n = T_nT_s$.

(c) [4 points] It is a fact that the T_s and T_n from part (b) can be expressed as polynomials in T with coefficients in $\mathbb{C}$. You don't need to prove this fact, but assuming it, prove that there is a *unique* decomposition $T = T'_s + T'_n$, where T'_s is diagonalizable, T'_n is nilpotent, and $T'_sT'_n = T'_nT'_s$.

4. Let $\mathbb{Q} \subset E$ be a finite Galois extension and let $B \subset E$ be the ring of algebraic integers in E. Suppose $P \subset B$ is a non-zero prime ideal with $P \cap \mathbb{Z} = (p)$, a prime ideal in $\mathbb{Z}$. Set $\overline{E} = B/P$ and suppose $\xi \in \overline{E}$ is a primitive generator for $\overline{E}$ over $\mathbb{F}_p = \mathbb{Z}/p$.

(a) [3 points] Explain why there exists $x \in B$ such that $\xi = x \mod P \in B/P = \overline{E}$ and such that $x \in \tau P \subset B$ for all $\tau \in Gal(E/\mathbb{Q})$ with $\tau P \neq P$.

(b) [7 points] If $G_P = \{\sigma \in Gal(E/\mathbb{Q}) \mid \sigma P = P\} \subset Gal(E/\mathbb{Q})$, prove that the obvious homomorphism $G_P \to Gal(\overline{E}/\mathbb{F}_p)$ is surjective.

5. Suppose that A is a noetherian integral domain. Suppose further that for every *maximal* ideal $Q \subset A$, the quotient Q/Q^2 is a one dimensional vector space over the field A/Q.

(a) [5 points] Prove that every non-zero prime ideal of A is maximal.

[Hint: Prove something about the localizations $A_{(Q)}$ for *maximal* ideals Q.]

(b) [5 points] Prove that A is integrally closed.

[In both (a) and (b), give precise statements of any lemmas you use.]