Stanford Mathematics PhD Qualifying Exam
Algebra – Spring 2007
Morning Session

1. Let V be a finite-dimensional vector space over a field F, and let S be a set of commuting linear transformations of V. Assume that for every $T \in S$ the characteristic polynomial of T factors into linear factors over F, not necessarily distinct. Prove that V has a basis with respect to which every $T \in S$ is represented by an upper-triangular matrix.

2. Let K be a field, p a prime. Let $\mu \subset K$ be the group of p-th roots of unity in K. Assume that $|\mu| = p$.

(i) Prove that the characteristic of K is not equal to p.

(ii) Let $\sigma: K \longrightarrow K$ be an automorphism of order p, and let F be the fixed field of σ. Show that $\mu \subset F$ and deduce that $K = F(\alpha)$ for some α such that $\alpha^p \in F$. State any theorems you quote.

3. Let A be a commutative ring and I an ideal. Recall that the *radical* $\sqrt{I}$ is the ideal $\{x \in A | x^n \in I \text{ for some } n\}$, and that I is called *primary* if whenever $ab \in I$ we have either $a \in I$ or $b \in \sqrt{I}$.

(a) Let A be a commutative ring, and let $I \in A$ be an ideal such that the radical $\sqrt{I} =$ is maximal. Prove that I is primary.

(b) Show that if $A = \mathbb{Q}[x, y]$ is a polynomial ring in two variables and $I = (x^3, x^2 + xy)$, then $\sqrt{I}$ is prime, but I is not primary.

4. Classify the finite groups of order $2007 = 3^2 \cdot 223$. (**Hint:** 223 is prime; $222 = 2 \cdot 3 \cdot 37$.)

5. Let G be the finite group of order 16 with generators and relations

$$G = \langle x, y | x^8 = y^2 = 1, yxy^{-1} = x^3 \rangle.$$

Find the conjugacy classes and construct the character table of G.

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Afternoon Session

1. Let G be a nonabelian group of order $117 = 3^2 \cdot 13$. Show that G has a normal cyclic subgroup of index 3. Find the degrees of the irreducible representations, and deduce number of conjugacy classes. (**Hint:** there are two possible G but the answer is the same for both. You can do both cases simultaneously.)

2. Let G be a finite group of odd order, and let p be the smallest prime dividing $|G|$. Suppose that G has a normal p-Sylow subgroup P of order $\leqslant p^2$. Prove that P is contained in the center of G.

3. Suppose that $A \subset B$ are integral domains so that the field of fractions of B is algebraic over the field of fractions of A.

(i) If $Q \subset B$ is a nonzero prime ideal, prove that $A \cap Q \neq 0$.

(ii) Assume that A is a principal ideal domain. Prove that every non-zero prime ideal of B is maximal.

4. Let F be a finite field with q elements, and let $W = F^6$. Count the number of pairs (U, V) where $W \supset V \supset U$, with U a 2-dimensional subspace and V a 4-dimensional subspace.

5. Let p be a prime, $a \in \mathbb{Q}$.

(a) Prove that either $x^p - a$ is irreducible, or it has a root in $\mathbb{Q}$. [**Hint:** what is the factorization of $x^p - a$ over $\mathbb{C}$?]

(b) Show that the splitting field of $x^p - a$ over $\mathbb{Q}$ contains no primitive p^2 root of 1.