Stanford Mathematics PhD Qualifying Exam
Algebra – Fall 2007
Morning Session

1. Let G be a group and X and Y two sets with (left) actions of G. We say the actions are *equivalent* if there is a bijection $\phi\colon X \longrightarrow Y$ such that $\phi(g \cdot x) = g \cdot \phi(x)$. Fix elements $x_0 \in X$ and $y_0 \in Y$. Let $H = \{g \in G \mid g \cdot x_0 = x_0\}$ and $K = \{g \in G \mid g \cdot y_0 = y_0\}$ denote the isotropy subgroups of x_0 and y_0, respectively.

(a) If the actions on X and Y are transitive, show the actions are equivalent if and only if H and K are conjugate.

(b) Let $F = \mathbb{F}_p$ where p is prime, and let $G = \mathrm{GL}(2, \mathbb{F}_p)$. Here are two sets X and Y with actions of G. The set X is the projective line, consisting of all one-dimensional subspaces of the two-dimensional vector space $\mathbb{F}_p^2$; and Y is the set of p-Sylow subgroups of G. Here the action of G on X is by matrix multiplication, and the action of G on Y is by conjugation. Show that these two actions are equivalent. [**Hint:** Let $V = F\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in X$. What is the isotropy subgroup of V?]

2. Show that there is a nonabelian group G of order $3 \cdot 13 = 39$. Describe G by generators and relations, find the conjugacy classes and construct the character table.

3. Suppose that A is a commutative Noetherian ring.

(a) Prove that every ideal $\mathfrak{a}$ of A contains a finite product $\mathfrak{p}_1 \cdots \mathfrak{p}_r$ of prime ideals.

(b) Prove that A has only finitely many minimal prime ideals, and that every prime ideal of A contains at least one of these.

4. Let $F = \mathbb{F}_p$ where p is prime. How many irreducible polynomials over F are there of degrees $2, 3$ and 6?

5. Consider the field of rational functions $E = \mathbb{Q}(s, t, u, v)$ in four variables. Let $F \subset E$ denote the fixed field of the obvious action of S_4 on E permuting $\{s, t, u, v\}$. Show that $w = st + uv \in E$ has degree 3 over F. What is the Galois group $\mathrm{Gal}(E/F(w))$?

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Afternoon Session

1. Classify all groups of order 225 up to isomorphism.

2. Let G be a finite group with center $Z(G)$. Show that the number of irreducible complex representations of G is at least $|Z(G)|$. (**Hint:** first prove that if θ: $Z(G) \longrightarrow \mathbb{C}^\times$ there is an irreducible complex representation π: $G \longrightarrow \mathrm{GL}(V)$ such that $\pi(zg) = \theta(z)\pi(g)$ for $z \in Z(G)$.)

3. (a) Suppose that g is a complex matrix such that $\mathbb{C}^n$ has a basis of eigenvectors $v_1, \cdots, v_n$ such that $v_i \in \mathbb{R}^n$ and the v_i are orthogonal with respect to the usual dot product, which is the symmetric bilinear form

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^n x_i y_i.$$

Prove that there exists a real orthogonal matrix $k = (k^t)^{-1}$ such that $kgk^{-1} = kgk^t$ is diagonal.

(b) Let $g \in \mathrm{GL}(n, \mathbb{C})$ be a unitary symmetric matrix; that is, $g = g^t$ and $g \cdot \bar{g}^t = I$. Prove that there exists a real orthogonal matrix $k = (k^t)^{-1}$ such that $kgk^{-1} = kgk^t$ is diagonal. (**Hint:** Let λ be an eigenvalue of g. Observe that $|\lambda| = 1$ and show that $V_\lambda = \{v \in \mathbb{C}^n | \, gv = \lambda g\}$ is stable under complex conjugation.)

4. (a) Suppose k is an infinite field, $k(\alpha, \beta)$ an algebraic extension of k such that β is separable over k (α is not assumed separable). Let $f(x)$ and $g(x)$ denote the minimal polynomials of α and β respectively over k. Show that there exists a $c \in k$ such that if $\theta = \alpha + c\beta$ then the polynomials $g(x)$ and $h(x) = f(\theta - cx)$ have exactly one root in common in an algebraic closure of k, namely $x = \beta$.

(b) Deduce that $k(\alpha, \beta) = k(\theta)$. [**Hint:** What is the greatest common divisor of the polynomials $g(x)$ and $h(x)$ in the polynomial ring $k(\theta)[x]$?]

5. Let A be an integral domain with field of fractions K.

(a) If $\mathfrak{m}$ is a maximal ideal of A then the localization $A_\mathfrak{m}$ can be regarded as a subring of K. Prove that

$$A = \bigcap_{\text{maximal } \mathfrak{m}} A_\mathfrak{m}.$$

(b) Show that if $A_\mathfrak{m}$ is a unique factorization domain for all maximal ideals $\mathfrak{m}$ then A is integrally closed in K.