Stanford Mathematics PhD Qualifying Exam

Algebra – Spring 2006

Morning Session

1. Suppose that H is a subgroup of a group G of index n. Show that G has a normal subgroup of index $\leqslant n!$. Use this to prove that there is no finite simple group of order $2430 = 2 \cdot 3^5 \cdot 5$.

2. Let G be the following group of order 28.

$$G = \langle x, y | x^7 = y^4 = 1, yxy^{-1} = x^{-1} \rangle.$$

Determine the conjugacy classes of G and compute its character table.

3. (i) Suppose that $d > 1$ is a square-free integer and $d \equiv 1$ modulo 4. Determine (with proof) the ring of algebraic integers in $\mathbb{Q}(\sqrt{d})$.

(ii) Explain how the principal ideals (2), (3) and (13) factor into prime ideals in the ring of algebraic integers in $\mathbb{Q}(\sqrt{13})$.

4. Find the Galois group of $x^4 + 1$ over $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$ and $\mathbb{F}_7$.

5. How many similarity classes are there of rational matrices with characteristic polynomial $(x^3 + 1)^3(x^2 + 1)^3$ and a minimal polynomial of degree 10?

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Afternoon Session

1. Let V be a finite-dimensional vector space over a field F of characteristic p and let T: $V \longrightarrow V$ be a linear transformation such that $T^p = I$ is the identity map.

(i) Show that T has an eigenvector in V.

(ii) Show that T is upper triangular with respect to a suitable basis of V.

2. Let G be a finite group, and let H be a subgroup of index two. Let $x \in H$ and let $C_G(x)$ be the centralizer of x in G.

(i) Prove that if $C_G(x) \not\subset H$ then the conjugacy class of x in G agrees equals the H-conjugacy class of x; on the other hand, if $C_G(x) \subset H$ then the conjugacy class of x in G is contained in H but splits into two H-conjugacy classes.

(ii) Let G be the symmetric group S_9 and H be A_9. Determine the G-conjugacy classes of even permutations that split into two conjugacy classes in H. **Hint:** there are two.

3. Let A be a Noetherian commutative ring containing a field k and an ideal I such that if $J = \sqrt{I}$ is the radical of I then A/J is a finite-dimensional k-vector space. Prove that A/I is also a finite-dimensional k-vector space.

4. Let G be a finite p-group, and $\lambda: G \longrightarrow \mathbb{C}^\times$ a homomorphism. Assume that the order of λ is a prime p, so that $H = \ker(\lambda)$ is a subgroup of index p. Let θ be an irreducible character of G such that $\lambda\theta = \theta$. Show that $\langle \theta, \theta \rangle_H = p$ and deduce that θ is induced from a character of H.

5. Let $\zeta = e^{2\pi i/7}$. Find an element α of $\mathbb{Q}(\zeta)$ such that $[\mathbb{Q}(\alpha) \colon \mathbb{Q}] = 3$. Show that there does not exist $\beta \in \mathbb{Q}(\alpha)$, $\beta \notin \mathbb{Q}$ such that $\beta^3 \in \mathbb{Q}$.