Stanford Mathematics PhD Qualifying Exam

Algebra – Fall 2006

Morning Session

1. In parts (a) and (c), let G be a nonabelian group of order 56.

(a) Prove that G has a normal 2-Sylow subgroup or a normal 7-Sylow subgroup.

(b) Let Z_n denote a cyclic group of order n. Compute the order of $\mathrm{Aut}(Q)$ when $Q = Z_8$, $Z_4 \times Z_2$ and $Z_2 \times Z_2 \times Z_2$.

(c) How many isomorphism classes are there of nonabelian groups of order 56 with normal abelian 2-Sylow subgroup? Explain. (**Hint:** use part (b).)

2. Let G be the following group of order 42.

$$G = \langle x, y \,|\, x^7 = y^6 = 1,\, yxy^{-1} = x^2 \rangle.$$

Determine the conjugacy classes of G and the degrees of its irreducible characters. Compute the values of at least one irreducible character of degree > 1.

3. Find an extension E of $\mathbb{Q}$ with $\mathrm{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$.

4. Suppose J is an $n \times n$ matrix over an algebraically closed field of characteristic $\neq 3$ and minimal polynomial $(T - \lambda)^n$ where $\lambda \neq 0$. Find the Jordan canonical form of J^3.

5. In this exercise, "commutative ring" means "commutative ring with unit," and it is assumed that a ring homomorphism $f: A \longrightarrow B$ satisfies $f(1_A) = 1_B$. If A, B and C are commutative rings, we say that C is a *coproduct* of A and B if there exist ring homomorphisms $\alpha: A \longrightarrow C$ and $\beta: B \longrightarrow C$ such that if D is any commutative ring, and $f: A \longrightarrow D$ and $g: B \longrightarrow D$ are ring homomorphisms, there exists a unique ring homomorphism $\phi: C \longrightarrow D$ such that $f = \phi \circ \alpha$ and $g = \phi \circ \beta$.

(a) If A and B are commutative rings, regard them as $\mathbb{Z}$-modules, and let $A \otimes B = A \otimes_{\mathbb{Z}} B$. Explain briefly why $A \otimes_{\mathbb{Z}} B$ naturally has the structure of a commutative ring.

(b) Prove that C is a coproduct of A and B if and only if $C \cong A \otimes B$.

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Afternoon Session

1. Let G be a p-group and H a nontrivial normal subgroup. Show that $H \cap Z(G)$ has at least p elements.

2. Let $A \subset B$ be finite abelian groups, and let $\chi\colon A \longrightarrow \mathbb{C}^\times$ be a homomorphism (linear character). Show that χ can be extended to B, and that the number of such extensions equals $[B\colon A]$.

3. Let $q = p^n$ where p is a prime, and let $\mathbb{F}_q$ denote the finite field with q elements. Show that the Frobenius automorphism $\sigma\colon \mathbb{F}_q \longrightarrow \mathbb{F}_q$ defined by $\sigma(x) = x^p$ is diagonalizable as a linear transformation *over* $\mathbb{F}_p$ if and only if n divides $p-1$.

4. Determine the splitting field K and Galois group $\mathrm{Gal}(K/\mathbb{Q})$ of the polynomial $x^4 - 2$ over the field $\mathbb{Q}$. Find all quadratic extensions of $\mathbb{Q}$ contained in K.

5. Let $F \subset K$ be fields. Let R be the polynomial ring $F[X]$, where X is an indeterminate, and similarly let $S = K[X]$.

(a) Show that if f and g are monic polynomials in S, and $S/fS \cong S/gS$ as S-modules, then $f = g$.

(b) Show that if $x \in R$ then $S \otimes_R (R/xR) \cong S/xS$.

(c) Suppose that M, N be finitely generated R-modules. Show that if $S \otimes_R M \cong S \otimes_R N$ as S-modules then $M \cong N$ as R-modules.