

ALGEBRA QUALIFYING EXAM, SPRING 2003: PART I

Directions: Work each problem in a separate bluebook. Give reasons for your answers, and make clear which facts you are assuming.

Notation:

\mathbb{Z} : Integers

\mathbb{Q} : Rational Field

\mathbb{R} : Real Field

\mathbb{C} : Complex Field

$\text{GL}(n, R)$: Group of invertible $n \times n$ matrices with entries in the ring R

\mathbb{F}_q : Finite field with q elements

\mathbb{Z}/n : Ring of integers mod n (can also be regarded as the cyclic group of order n)

1. Classify all finite groups of order 140 up to isomorphism.
2. Find the degree $|E : \mathbb{Q}|$ if E is the splitting field of the polynomial $X^{10} - 5 \in \mathbb{Q}[X]$. How many distinct intermediate fields K exist with $\mathbb{Q} \subsetneq K \subsetneq E$?
- 3(a). Find all positive integers that can occur as the order of some element of $\text{GL}(2, \mathbb{R})$. Exhibit an element of order 5.

(b). Find all positive integers that can occur as the order of some element of $\text{GL}(3, \mathbb{F}_7)$.

(c). Find all positive integers that can occur as the order of some element of $\text{GL}(4, \mathbb{Q})$. Exhibit an element of order $\neq 1$ or 2.
- 4(a). Find the integral closure B of the integers \mathbb{Z} in the field $\mathbb{Q}[\sqrt{-39}]$.

(b). Show that there are two distinct prime ideals of B that contain the ideal $5B \subset B$. Give generators for these two prime ideals and show that neither is principal.

(c). How does the ideal $3B \subset B$ factor as a product of prime ideals?
5. Let $\rho : G \rightarrow \text{GL}(3, \mathbb{C})$ be a 3-dimensional complex representation of a finite group G . Let V be the vector space of all 3×3 matrices over \mathbb{C} . Define the adjoint representation $\widehat{\rho} : G \rightarrow \text{GL}(V)$ by
$$\widehat{\rho}(g)A = \rho(g) A \rho(g^{-1})$$
for $g \in G$ and $A \in V$. Which integers can occur as the multiplicity of the trivial one dimensional representation in $\widehat{\rho}$?

ALGEBRA QUALIFYING EXAM, SPRING 2003: PART II

Directions: Work each problem in a separate bluebook. Give reasons for your answers, and make clear which facts you are assuming.

Notation:

\mathbb{Z} : Integers

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$: Fields of rational, real, and complex numbers, respectively

$\text{GL}(n, R)$: Group of $n \times n$ invertible matrices with entries in the ring R

\mathbb{F}_q : Finite field with q elements

\mathbb{Z}/n : Ring of integers mod n (can also be regarded as the cyclic group of order n)

1(a). Find all abelian groups G that contain a subgroup H isomorphic to $\mathbb{Z}/72\mathbb{Z}$ for which the quotient group G/H is also isomorphic to $\mathbb{Z}/72\mathbb{Z}$.

(b). Find the invariant factors of the abelian group $(\mathbb{Z}/44,000\mathbb{Z})^*$, i.e., the multiplicative group of invertible elements in the ring $\mathbb{Z}/44,000\mathbb{Z}$.

2. Suppose $I \subset \mathbb{Q}[x_1, x_2, \dots, x_n]$ is an ideal such that the set of zeroes

$$V(I) = \{\mathbf{x} \in \mathbb{C}^n : f(\mathbf{x}) = 0 \text{ for all } f \in I\}$$

is a finite set. Show that the ring $\mathbb{Q}[x_1, x_2, \dots, x_n]/I$ is a finite dimensional vector space over \mathbb{Q} .

[Hint: first show that $\mathbb{Q}[x_1, x_2, \dots, x_n]/J$ is finite dimensional, where $J = \sqrt{I}$ is the radical of I . Then consider powers $J \supset J^2 \supset \dots$]

3(a). Factor $X^5 + 7X^3 + 6X^2 + X + 5$ over the fields $\mathbb{F}_2, \mathbb{F}_3$, and \mathbb{F}_5 .

[You may assume the (true) result that this polynomial has no irreducible quadratic factors over \mathbb{F}_3 .]

(b). What are the Galois groups of $X^5 + 7X^3 + 6X^2 + X + 5$ over $\mathbb{F}_2, \mathbb{F}_3$, and \mathbb{F}_5 ?

(c). What are the Galois groups of $X^5 + 7X^3 + 6X^2 + X + 5$ over \mathbb{Q} ?

4. Suppose V is a finite dimensional vector space over a field k and suppose $A : V \rightarrow V$ is a k -linear endomorphism whose minimal polynomial is *not* equal to its characteristic polynomial. Show that there exist k -linear endomorphisms $B, C : V \rightarrow V$ such that $AB = BA, AC = CA$, but $BC \neq CB$.

5(a). Produce a complex character table for the symmetric group S_4 .

(b). The rotation group of the cube is isomorphic to S_4 as a permutation group of the four diagonals of the cube. Let $\rho : S_4 \rightarrow \text{GL}(8, \mathbb{C})$ be the permutation representation of S_4 defined by the action of the rotation group on the eight vertices of the cube. Find the character of ρ .

(c). Decompose ρ as a direct sum of irreducible representations of S_4 .