

## ALGEBRA QUAL, SPRING 2008, PART II

1(a). Let  $H$  be a non-abelian group of order 20. Determine all possible pairs  $\{(n_5, n_2)\}$ , where  $n_5$  and  $n_2$  denote the number of Sylow-5 and Sylow-2 subgroups in such an  $H$ .

1(b). Let  $G$  be a group of order 60 that contains a non-abelian subgroup of order 20. Determine all possible pairs  $\{(m_5, m_3)\}$ , where  $m_5$  and  $m_3$  denote the number of Sylow-5 and Sylow-3 subgroups in such a  $G$ .

[Hint: Normalizers. Use part (a).]

2. Let  $G$  be a group. Let  $Aut(G)$  denote the group of automorphisms of  $G$  and let  $Inn(G)$  denote the group of inner automorphisms of  $G$ .

(a). Show that  $Inn(G)$  is a normal subgroup of  $Aut(G)$ .

(b). Determine the group  $Inn(S_4)$ , where  $S_4$  is the symmetric group on four letters.

(c). Show that all automorphisms of  $S_4$  are inner.

[Hint: Conjugacy classes. Show that an automorphism must map a 2-cycle to a 2-cycle. Then do some counting.]

3(a). Suppose  $A \subset B = A[x_1, \dots, x_n]$  are integral domains so that as a ring  $B$  is finitely generated over  $A$ , with generators  $x_i$  algebraic over  $A$ . Explain why there exists a non-zero element  $a \in A$  so that every element of  $B[1/a]$  is integral over  $A[1/a]$ .

3(b). Suppose  $A \subset B$  is an integral extension of commutative rings and suppose  $K$  is an algebraically closed field. Explain why every homomorphism  $\phi: A \rightarrow K$  extends to a homomorphism  $\Phi: B \rightarrow K$ .

4(a). Determine the Galois group of  $X^5 - 2$  over the fields  $\mathbb{R}, \mathbb{F}_7$ , and  $\mathbb{F}_{11}$ .

4(b). If  $E$  is the splitting field of  $X^5 - 2$  over  $\mathbb{Q}$ , describe the fixed fields  $L_i$  of all Sylow-2 subgroups  $H_i \subset Gal(E/\mathbb{Q})$ .

5(a). If  $p$  is a prime integer, how many irreducible monic polynomials of degree 3 are there over the field  $\mathbb{F}_p$ ?

5(b). How many conjugacy classes are there in the group  $GL(3, \mathbb{F}_p)$ ?

[Hint: Use linear algebra, specifically invariant factors.]