ALGEBRA QUAL, SPRING 2008, PART I

1(a). In both the dihedral group of order 8 and the quaternion group of order 8, determine how many subgroups there are of order 4. In both these groups, $G = D_8$ or Q_8, determine the number of $Aut(G)$ orbits in the set of subgroups of order 4.

[The action is the obvious one, $H \mapsto \phi H$, for $H \subset G$ and $\phi \in Aut(G)$.]

1(b). Determine the isomorphism classes of groups of order 2008 that have a non-abelian Sylow-2 subgroup.

[The odd factor of 2008 is prime.]

2. Suppose k is a field and suppose $E \supset k$ is an integral domain that is finite dimensional as a vector space over k.

(a). Explain why E is a field.

(b). If $a \in E$, consider the k-linear transformation $\alpha\colon E \to E$ defined by $\alpha(x) = ax$. In terms of the minimal polynomial $f(T) \in k[T]$ of a over k, describe the rational canonical form of α.

(c). If $a \in E$ is purely inseparable over k, that is, if its minimal polynomial has form $f(T) = T^{(p^e)} - r$, where $r \in k,\ p = char(k)$, describe the Jordan canonical form of α over the algebraic closure of k.

3(a). Let A be a commutative Noetherian ring and suppose $I \subset A$ is an ideal so that every prime ideal $P \supset I$ is maximal. Explain why A/I is isomorphic to a direct product $A_1 \times \cdots \times A_n$, where each ring A_j has a unique prime ideal.

3(b). Suppose $(0) = \bigcap_{1 \le i \le r} Q_i$ is an irredundant primary decomposition of the zero ideal in a commutative ring B. Let $P_i = \sqrt{Q_i}$. Show that the set of zero divisors in B is exactly $\bigcup_{1 \le i \le r} P_i$.

[A *zero divisor* is an element $b \in B$ so that $ab = 0$ for some $a \neq 0$. *Irredundant* includes the fact that $\bigcap_{j \neq i} Q_j \not\subset Q_i$.]

4. Let A be a Dedekind domain with field of fractions K. Let $E \supset K$ be a finite Galois extension, $G = Gal(E/K)$. Let $B \supset A$ be the integral closure of A in E.

(a). Explain why the map from the set of ideals in A to the set of ideals in B given by $I \mapsto IB$ is injective.

[Hint: You should use facts about factorization of ideals in Dedekind domains.]

Define an operation, N, on ideals $J \subset B$ by

$$N(J) = \prod_{\sigma \in G} \sigma J \subset B.$$

(b). If $I, J \subset B$ are ideals, show $N(IJ) = N(I)N(J)$. If $b \in B$ and $n = N_{E/K}(b) \in A$ is the field norm of b, show that $N(bB) = nB$, where bB and nB denote principal ideals.

(c). If $Q \subset B$ is a prime ideal and $P = Q \cap A$, let $f = |B/Q \; : \; A/P|$. Show that $N(Q) = (PB)^f$.

[Hint: Make use of the factorization of the ideal PB in B.]

(d). If $J \subset B$ is an ideal so that $N(J) = PB$ for some prime ideal $P \subset A$, then J is a prime ideal of B.

5(a). Let $\zeta_7, \; \zeta_9 \in \mathbb{C}$ denote primitive 7^{th} and 9^{th} roots of unity, respectively. Find the Galois group of $E = \mathbb{Q}[\zeta_7, \; \zeta_9]$ over $\mathbb{Q}$.

5(b). Let $\rho_7, \; \rho_9 \in \mathbb{R}$ denote the real parts of ζ_7 and ζ_9, respectively. Explain why $L = \mathbb{Q}[\rho_7, \; \rho_9]$ is normal over $\mathbb{Q}$, and determine the Galois group of L over $\mathbb{Q}$.